# FISIKA KUANTIKOA 2. KUATRIA

## (2. partea)

## PARTIKULA BEREIZTEZINAK eta ATOMO ELEKTROIANITZAK

17-03-24

### Partikula bat baino gehiago:

Sisteman dauden elektroi guztiak hartu behar dira kontuan → Sistema deskribatzen duen hamiltondarrak haien ezaugarriak barneratu beharko ditu.

→ orokorrean bi partikulen posizioen menpekoa

- 2 partikula badira ⇒ * $H(1,2) = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2)$

  $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1) + V(\vec{r}_2)$ ⇒ bi partikulak independenteak dira (elkarrekintzarik ez)

*Sistema deskribatzeko uhin funtzioa → bi partikulen ezaugarriak:

* $\psi(\vec{r}_1, \vec{r}_2)$ ⇒ normalizatzeko ⇒ $(\psi, \psi) = 1$ →

- $(\psi, \psi) = \int \psi^*(\vec{r}_1, \vec{r}_2)\, \psi(\vec{r}_1, \vec{r}_2)\, d^3\vec{r}_1\, d^3\vec{r}_2 = 1$

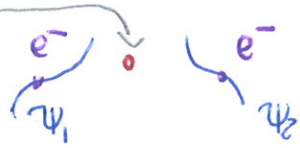
- $\langle \vec{p}_1 \rangle_\psi = (\psi, \hat{\vec{p}}_1 \psi) = -i\hbar(\psi, \vec{\nabla}_1 \psi) = -i\hbar \int \psi^*(\vec{r}_1, \vec{r}_2)\, \vec{\nabla}_1 \psi(\vec{r}_1, \vec{r}_2)\, d^3\vec{r}_1\, d^3\vec{r}_2$

### Partikula bereizgarriak eta bereiztezinak.
### (Mekanika klasikoa eta kuantikoaren ikuspuntutik)

Demagun 2 partikula ditugula → elektroia eta positroia (Karga ezberdina, baina spin eta masa berdina) ⇒ bereizgarriak $e^-$ $e^+$ haien propietateak kontuan mekanika klasiko zein kuantikoan ⇒ Partikulak ezberdinak badira beti izango dira bereizgarriak.

• Partikulak berdinak badira ⇒  Mekanika Klasikoan bakoitzaren kokatzeko ibilbide bat definitu dezakegu beti, $\vec{r}_1(t)$ eta $\vec{r}_2(t)$ ⇒ bakoitza non dagoen esan dezakegu eta ezaugarri fisikoak berdinak izan arren biak bereizi ditzakegu.

Mekanika Kuantikoan ez dago aukeran ibilbidea zehazterik ⇒ partikulek uhin funtzio bat izan dezakete haiekin lotuta baina honek ez dute zehazten adierazten partikulen posizioa; dentsitate probabilitatea baizik. Zenbait puntutan, biek puntu horretan espazio probabilitatea izan dezakete beraz ezin dugu zehaztu puntu horretan ikusi dugun partikula.




Ziurgabetasun hau ⇒ Heissenberg → posizioa ez dago guztiz zehaztuta; uhin funtzioek badute zabalera bat eta zabalera hori dela eta bi uhin-funtzio solapatzen badira bi partikula horiek bereiztezinak izango dira mekanika kuantikoan ikuspuntutik (Atomo batean adibidez)

## Partikula bereiztezinen uhin-funtzioak: simetrikoak (bosoiak) edo antisimetrikoak (fermioiak).

Partikulak bereiztezinak izateak baldintza bat jartzen dute uhin-funtzioetan; hauek ezin dira edozein izan, ezaugarriren bat izan behar dute.

• Adibidez, 1 eta 2 partikulak baditugu ⇒ $\Psi(\vec{r}_1, \vec{r}_2)$ beste zenbaki kuantiko batzuk izan ditzakegu ($m_s$, $s$, ...)

Bereiztezinak izateak, $\vec{r}_1$ eta $\vec{r}_2$ aldatzean $\Psi(\vec{r}_2, \vec{r}_1)$-ek aurrekoaren informazio fisiko berdina izan behar du.

$\hat{T}_{12}$ eragilea ; truke eragilea → bi partikulak trukatu: $\hat{T}_{12}\psi(\vec{r}_1,\vec{r}_2)=\psi(\vec{r}_2,\vec{r}_1)$

Informazio fisikoa berdina izan behar bada ere, $\psi(\vec{r}_2,\vec{r}_1)$-k ez du $\psi(\vec{r}_1,\vec{r}_2)$-ren

berdina izan behar, edozein uhin-funtzioak modulu bateko zenbaki induktiboa batekin

biderkatzean informazio berdina duelako ⇒ orduan $\psi(\vec{r}_2,\vec{r}_1)=e^{i\gamma}\psi(\vec{r}_1,\vec{r}_2)$

Printzipioz $\gamma$ edozein izan daiteke eta horren arabera $\psi(\vec{r}_1,\vec{r}_2)$ $\hat{T}_{12}$-ren

autobektoreak izango dira, autobalioak $e^{i\gamma}$ izanik.

Hala ere, $\hat{T}_{12}$ hermitikoa denez autobalioak errealak izango dira, beraz, $\gamma=0,\pi$

$$\psi(\vec{r}_2,\vec{r}_1)=\pm\psi(\vec{r}_1,\vec{r}_2)$$

Halaber $\hat{T}_{12}(\hat{T}_{12}\psi(\vec{r}_1,\vec{r}_2))=\hat{T}_{12}(e^{i\gamma}\psi(\vec{r}_2,\vec{r}_1))=e^{i\gamma}\hat{T}_{12}\psi(\vec{r}_2,\vec{r}_1)=e^{2i\gamma}\psi(\vec{r}_1,\vec{r}_2)$

eta $\hat{T}_{12}\hat{T}_{12}=\mathbb{1}$ denez ⇒ $e^{2i\gamma}\psi(\vec{r}_1,\vec{r}_2)=\psi(\vec{r}_1,\vec{r}_2)$ → $e^{2i\gamma}=1$ →

$$e^{i\gamma}=\pm1 \rightarrow \gamma=0,\pi$$

Hau da, $\psi(\vec{r}_1,\vec{r}_2)$ simetrikoa edo antisimetrikoa izango da.

1. Simetrikoa ⇒ $\psi(\vec{r}_1,\vec{r}_2)=\psi(\vec{r}_2,\vec{r}_1)$
2. Antisimetrikoa ⇒ $\psi(\vec{r}_1,\vec{r}_2)=-\psi(\vec{r}_2,\vec{r}_1)$

- Simetrikoak diren partikulak bosoiak dira ⇒ spin osoa dute, $s\in\mathbb{N}$
- Antisimetrikoak diren partikulak fermioiak dira ⇒ spin erdi osoa dute

## Truke-endekapena eta Pauli-ren ezberdintasunaren printzipioa.

Bi fermioi baditugu eta haiek bereiztezinak badira sistema deskribatzeko

dugun uhin-funtzioa derrigorrez antisimetrikoa izango da, eta bosoiak badira simetrikoak. Beste ikuspuntu batetik aztertuz,

- Bi partikula $\Rightarrow$ $\hat{H}_{(1,2)} \xrightarrow{\text{bereiztezinak}} \hat{H}_{(1,2)} = \hat{H}_{(2,1)}$
- Truke eragilea $\rightarrow$ $\hat{T}_{12}\hat{H}_{(1,2)} = \hat{H}_{(2,1)}\hat{T}_{12} = \hat{H}_{(1,2)}\hat{T}_{12}$ $\Rightarrow$ Trukakorrak

Beraz, $\hat{T}_{12}$-ren autofuntzioak simetrikoak eta antisimetrikoak direnez (eta autobalioak $\pm 1$) $\hat{H}_{(1,2)}$-ren autofuntzioak beti aukera daitezke antisimetrikoak edo simetrikoak izatea (trukakorrak izatean bien aldibereko oinarri bat sortu daitekeelako):

* Partikulak bosoiak badira sistema horren uhin-funtzioak <u>simetrikoak</u> izatea derrigortu behar dugu.

* Partikulak fermioiak badira sistema horren uhin-funtzioak <u>antisimetrikoak</u> izatea derrigortu behar dugu.

- Hemendik Pauliren elkarrezintasun printzipioa ondorioztatzen dugu:

Fermioiak baditugu, bi elektroi aldiberez, haien zenbaki kuantiko <u>guztiak</u> ezin dira berdinak izan. (Bosoien kasuan berdinak izan daitezke)

Demagun zenbaki kuantiko berdinak dituztela bi elektroiek:

zenbaki kuantiko guztiak

$$\psi_n(x_1)\,\psi_n(x_2) = \psi(x_1,x_2) = -\psi(x_2,x_1) = -\psi_n(x_2)\,\psi_n(x_1)$$

$\downarrow$

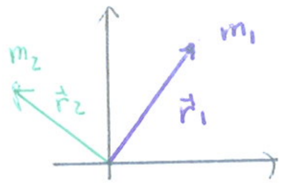
Fermioiak

* Aukera bakarra $\psi_n(x_1)\,\psi_n(x_2) = 0$ izatea da $\rightarrow$ uhin funtzioa ezin denez nulua izan hasierako hipotesia <u>ez</u> da zuzena $\Rightarrow$ zenbaki kuantikoetako bat gutxienez ezberdina izan behar da.

Elektroiak mailetan banatzean, maila berdinean daudenean beti jartzen dugu aurkako spinekin. $\uparrow\downarrow$ $\Rightarrow$ ez dira $e^-$ gehiago sartzen $\rightarrow$ bestela $n$ berdina

# PARTIKULA INDEPENDENTEAK:

Partikula independenteak ⟷ partikulen artean elkarrekintzarik ez




Demagun bi partikula baino ez ditugula ($\vec r_1$, $\vec r_2$ posizioetan)

- Sistema deskribatzeko uhin-funtzioa ⇒ $\psi(\vec r_1, \vec r_2)$

- Helburua ⇒ Hamiltondarraren autofuntzioak kalkulatzea: $\hat H \psi(\vec r_1, \vec r_2) = \varepsilon \psi(\vec r_1, \vec r_2)$

Partikulak independenteak direnez hamiltondarra banangarria izango da: $\hat H_1 + \hat H_2$

$$(\hat H_1 + \hat H_2)\psi = \varepsilon \psi = \left( -\frac{\hbar^2}{2m_1} \nabla_1^2 + V(\vec r_1) - \frac{\hbar^2}{2m_2} \nabla_1^2 + V(\vec r_2) \right) \psi$$

$\vec r_1$ eta $\vec r_2$ banandurik agertzen dira ⇒ elkarrekintzarik <u>ez</u> dutelako.

- Hamiltondarra banangarria izanda aldagaien banatzea aplika daiteke: $\psi(\vec r_1, \vec r_2) = \psi_1(\vec r_1)\psi_2(\vec r_2)$

$\psi_1$-ekin $i$ zenbaki kuantikoa eta $\psi_2$-rekin $j$ zenbaki kuantikoa.

zenbaki kuantikoak

$\hat H_1$-ek $\psi^1$-en eragin bakarrik

$$(\hat H_1 + \hat H_2)\psi = (\hat H_1 + \hat H_2)\left(\psi_i^1(\vec r_1)\,\psi_j^2(\vec r_2)\right) = \varepsilon_i\, \psi_i^1(\vec r_1)\psi_j^2(\vec r_2) + \varepsilon_j\, \psi_i^1(\vec r_1)\psi_j^2(\vec r_2) =$$

$$\varepsilon_{i,j}\, \psi_i^1(\vec r_1)\psi_j^2(\vec r_2) \xrightarrow{\cdot 1/\psi} \frac{(\hat H_1 \psi_i^1(\vec r_1))\cancel{\psi_j^2(\vec r_2)}}{\psi_i^1(\vec r_1)\cancel{\psi_j^2(\vec r_2)}} + \frac{(\hat H_2 \psi_j^2(\vec r_2))\cancel{\psi_i^1(\vec r_1)}}{\cancel{\psi_i^1(\vec r_1)}\psi_j^2(\vec r_2)} =$$

$$\frac{\hat H_1 \psi_i^1}{\psi_i^1} + \frac{\hat H_2 \psi_j^2}{\psi_j^2} = \varepsilon_{ij} = \varepsilon_i + \varepsilon_j \Rightarrow \begin{cases} \dfrac{\hat H_1 \psi_i^1}{\psi_i^1} = \varepsilon_i & (1) \\[2ex] \dfrac{\hat H_2 \psi_j^2}{\psi_j^2} = \varepsilon_j & (2) \end{cases}$$

(1) ⇒ $\hat H_1 \psi_i^1 = \varepsilon_i\, \psi_i^1$

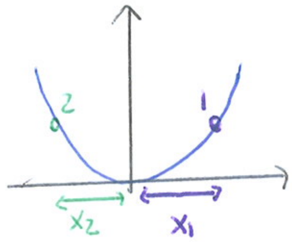
(2) ⇒ $\hat H_2 \psi_j^2 = \varepsilon_j\, \psi_j^2$

Partikula indibidualaei dagozkien autobalio eta autobektoreen problema. Hemendik sistema osoaren uhin-funtzioa lortu

- $\psi_{i,j}(\vec r_1, \vec r_2) = \psi_i^1(\vec r_1)\,\psi_j^2(\vec r_2) = \psi_i^1(\vec r_1) \otimes \psi_j^2(\vec r_2)$ , $\varepsilon_{ij} = \varepsilon_i + \varepsilon_j$

Dirac-en notazioan ⇒ $|i,j\rangle = |i\rangle \otimes |j\rangle$ Biderkadura tentsoriala

# PARTIKULA INDEPENDENTE eta BEREIZGARRI, $s=0$ (Adibiderik sinpleena)



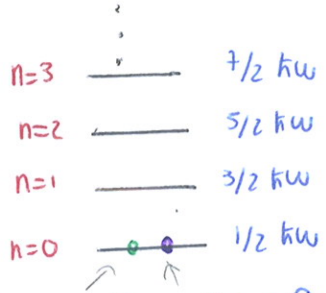

• Demagun bi partikula ditugula, independenteak, bereizgarriak eta spinik gabekoak. Halaber, bi partikulak osziladore harmonikoan daude: $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx_1^2} + \frac{1}{2}kx_1^2 - \frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + \frac{1}{2}kx_2^2$

2 partikula

$|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$ ; $\varepsilon_{n_1 n_2} = \varepsilon_{n_1} + \varepsilon_{n_2} = \left(\frac{1}{2}+n_1\right)\hbar\omega + \left(\frac{1}{2}+n_2\right)\hbar\omega$ ; $\omega = \sqrt{\frac{k}{m}}$

1. partikula

• Partikula independenteen energiak:




* Oinarrizko egoeran → energiarik baxuena, $n_1 = n_2 = 0 \rightarrow n = n_1 + n_2 = 0$ ; $\varepsilon_0 = \hbar\omega$

• $\psi_0 = \psi_0(x_1)\,\psi_0(x_2) \Rightarrow |0,0\rangle = |0\rangle \otimes |0\rangle$ (Dirac-en notazioan)

* Lehenengo egoera kitzikatua → partikula bat $n=0$-n eta bestea $n=1$-n → bi aukera: $(n_1=0,\ n_2=1)$, $(n_1=1,\ n_2=0)$ → $\varepsilon_1 = 2\hbar\omega$ ($g_1 = 2$ → endekapena)

• $\psi_1 = \psi_0(x_1)\,\psi_1(x_2)$ edo $\psi_1 = \psi_1(x_1)\,\psi_0(x_2) \Rightarrow$

• $|1,0\rangle = |1\rangle \otimes |0\rangle$ edo $|0,1\rangle = |0\rangle \otimes |1\rangle$ (Dirac)

* Bigarren egoera kitzikatua → $n_1 = 0,\ n_2 = 2$ ; $n_1 = 2,\ n_2 = 0$ ; $n_1 = n_2 = 1$

$\varepsilon_2 = 3\hbar\omega$ ($g_2 = 3$ → endekapena)

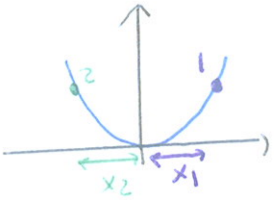
• $\psi_2(x_1, x_2) = \psi_0(x_1)\,\psi_2(x_2)$ edo $\psi_2(x_1,x_2) = \psi_2(x_1)\,\psi_0(x_2)$

edo $\psi_2(x_1,x_2) = \psi_1(x_1)\,\psi_1(x_2)$

• $|0,2\rangle = |0\rangle \otimes |2\rangle$ ; $|2,0\rangle = |2\rangle \otimes |0\rangle$ ; $|1,1\rangle = |1\rangle \otimes |1\rangle$

Hemendik beste maila kitzikatuak kalkulatu daitezke, prozedura bera jarraituz.

# PARTIKULA INDEPENDENTE eta BEREIZGARRIAK, $S=1/2$




Spinaren eragina H-n ez da sartzen

• Demagun bi partikula independente, bereizgarri eta $s=1/2$ spinekoak ditugula osziladore harmoniko batean:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx_1^2} + \frac{1}{2}Kx_1^2 - \frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + \frac{1}{2}Kx_2^2$$

• Ezberdintasun bakarra $\Rightarrow$ egoera zehazteko spin-egoera zehaztu behar da.

* $|n_1, m_{s_1}; n_2, m_{s_2}\rangle = |n_1, m_{s_1}\rangle \otimes |n_2, m_{s_2}\rangle$ — Spinaren osagaiak

* $\varepsilon_{n_1,n_2} = \varepsilon_{n_1} + \varepsilon_{n_2}$ * $\psi_{n_1,m_{s_1};n_2,m_{s_2}}(x_1,x_2) = \psi_{n_1}(x_1)\chi_1^{m_{s_1}} \otimes \psi_{n_2}(x_2)\chi_2^{m_{s_2}}$

• Partikula bakarrari dagozkion energia-mailak:

- $n=3$ — $7/2\,\hbar\omega$
- $n=2$ — $5/2\,\hbar\omega$
- $n=1$ — $3/2\,\hbar\omega$
- $n=0$ — $1/2\,\hbar\omega$

* Oinarrizko egoera $\to n_1=n_2=0 \to \varepsilon_0 = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{2} = \hbar\omega$

Baina spin egoera definitu behar da $\to m_{s_1} = \pm 1/2$, $m_{s_2} = \pm 1/2$

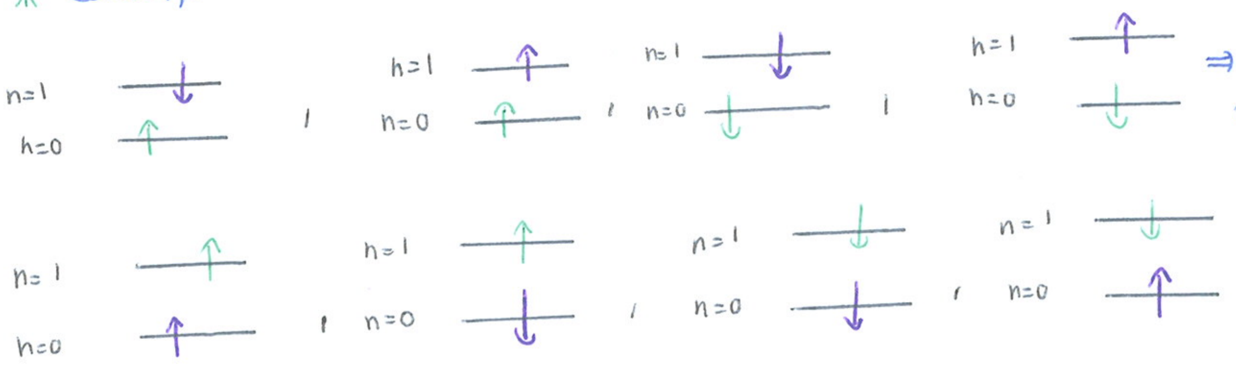
↗ bereizgarriak direnez spin paraleloa izan dezakete


$\Rightarrow g_0 = 4 = 2^2$

$|0,+;0,+\rangle$ , $|0,+;0,-\rangle$ , $|0,-;0,+\rangle$ , $|0,-;0,-\rangle$ (Dirac)

$\psi_{0,+;0,+}(x_1,x_2) = \psi_0(x_1)\chi_1^+\psi_0(x_2)\chi_2^+$ (Gauza bera beste egoerekin)

* Lehenengo maila kitzikatua $\to n_1=1, n_2=0$ edo $n_2=1, n_1=0 \to \varepsilon_1 = 2\hbar\omega$

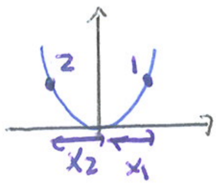



$\Rightarrow |1,-;0,+\rangle$

$\psi_{1,-;0,+}(x_1,x_2) = \psi_1(x_1)\chi_1^-\psi_0(x_2)\chi_2^+$

$g_1 = 8$

# PARTIKULA INDEPENDENTE eta BEREIZTEZINAK, $S=1/2$




- Demagun bi partikula ditugula, independenteak, bereiztezinak eta $S=1/2$ spindunak potentzial harmoniko batean:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx_1^2} + \frac{1}{2}Kx_1^2 - \frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + \frac{1}{2}Kx_2^2$$

- Egoerak bereizteko → zenbaki kuantikoak: $|n_1, m_{s_1}; n_2, m_{s_2}\rangle$

Bereiztezinak → ezin dezakegu jakin zein partikula dagoen egoera bakoitzean → uhin-funtzioa antisimetrikoa ($S=1/2 \notin \mathbb{N}$ delako)

Energia mailak ez dira aldatzen $\Rightarrow$ $E_{n_1,n_2} = (1+n_1+n_2)\hbar\omega$ $\quad \omega = \sqrt{\frac{K}{m}}$

- Energia mailak (partikula bakoitzarena):

$n=2$ ——— $5/2\ \hbar\omega$
$n=1$ ——— $3/2\ \hbar\omega$
$n=0$ ——— $1/2\ \hbar\omega$

* Oinarrizko egoera → energiarik baxuena → $E_0 = \hbar\omega$ → Aukera ezberdinak.

1) $n=0$ $\uparrow\uparrow$ 2) $n=0$ $\downarrow\downarrow$ 3) $n=0$ $\uparrow\downarrow$ → ordenak ez du inporta, ez da bereizten

1) $|0,+\rangle \otimes |0,+\rangle$

↳ $\psi_0(x_1)\chi_1^+\psi_0(x_2)\chi_2^+$ $\Rightarrow$ simetrikoa → Antisimetrikoa izan behar da

orduan egoera hau ezinezkoa da $\Rightarrow$ Paulíren esklusio printzipioa

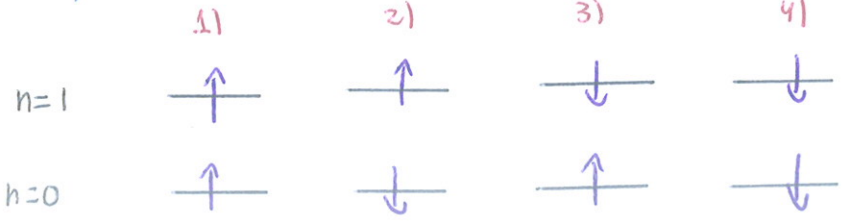
2) $|0,-\rangle \otimes |0,-\rangle$ → $\psi_0(x_1)\chi_1^-\psi_0(x_2)\chi_2^-$ $\Rightarrow$ simetrikoa → Antisimetrikoa izan behar da $\Rightarrow$ orduan egoera ezinezkoa.

Aukera bakarra $\uparrow\downarrow$ $\Rightarrow$ 3) $|0,+\rangle \otimes |0,-\rangle$ → Antisimetrizatu → $|0,-\rangle \otimes |0,+\rangle$ elkarpena ere kontsideratu behar da:

3) $\frac{1}{\sqrt{2}}\left[|0,+\rangle \otimes |0,-\rangle - |0,-\rangle \otimes |0,+\rangle\right]$ → uhin-funtzioa

$$\frac{1}{\sqrt{2}}\left(\psi_0(x_1)\chi_1^+\psi_2(x_2)\chi_2^- - \psi_0(x_1)\chi_1^-\psi_2(x_2)\chi_2^+\right)$$

* Lehenengo egoera kitzikatua → $\varepsilon_1 = 2\hbar\omega$ → Aukera ezberdinak. (Hemen partikula bakoitza energia maila ezberdinean dagoenez Pauliren esklusio printzipioak ez gaitu arduratuko)




Bi elektroiak kontuan hartu antisimetrizatuko → $n_1=0,\ n_2=1$

edo $n_2=0,\ n_1=1$

1) $\frac{1}{\sqrt{2}}\left[|0,+\rangle \otimes |1,+\rangle - |1,+\rangle \otimes |0,+\rangle\right]$ edo $\frac{1}{\sqrt{2}}\left[\psi_0(x_1)\chi_1^+\psi_1(x_2)\chi_2^+ - \psi_1(x_1)\chi_1^+\psi_0(x_2)\chi_2^+\right]$

Hau guztietan egin ⇒ Aukerak murriztu egin dira $g = 8 \to 4$

## ESPIN-EGOERA eta ANTISIMETRIZAZIOA:

$$\Psi = \frac{1}{\sqrt{2}}\left[\psi_0(x_1)\chi_1^+\psi_1(x_2)\chi_2^- - \psi_1(x_2)\chi_1^-\psi_0(x_2)\chi_2^+\right]$$

- Hamiltondarraren egoerek ez dute spin-egoera zehaztuta. Adibidez, demagun goiko egoera dugula, dinamikoko egoeraren soluzioetako bat (koeitzienteak → antisimetrikoa).

- Egoera hau ez da $S_{1z}$-ren egoera → ez dago zehaztuta $m_{S_{1z}}$.

  $S_{1z}$-ren egoera izateko ⇒ $\hat{S}_{1z}\Psi = a\Psi$

* Frogatu → (Gogoratu $\chi_1^+$ eta $\chi_1^-$-ren gainean baino ez duela eragingo)

$$S_{1z}\Psi = \frac{1}{\sqrt{2}}\left[\frac{\hbar}{2}\psi_0(x_1)\chi_1^+\psi_1(x_2)\chi_2^- + \frac{\hbar}{2}\psi_1(x_2)\chi_1^-\psi_0(x_2)\chi_2^+\right] \neq a\Psi$$

* Ez da $S_{1z}$-ren autofuntzioa!

* $[\hat{S}_{1z}, \hat{P}_{12}] \neq 0$ ⇒ Ez dugu izaten bien aldibereko autofuntzioak aurkitu

Batzuk itxaso dira → baina horretarako ezinbestean bi partikulen spin-egoera berdina izan behar da

- Gure egoerak onargarriak dira baina spin-egoera ez dago zehaztuta (partikula bakoitzarena) ⇒ spin osoa, ordea, bai.

$$\hat{\vec{S}} = \vec{S}_1 + \vec{S}_2 \quad ; \quad \hat{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 \Rightarrow [\hat{S}^2, \hat{T}_{12}] = 0 \text{ baita}$$

$$\hat{S}_z = S_{1z} + S_{2z} \Rightarrow [\hat{S}_z, \hat{T}_{12}] = 0 \quad \text{Spin osoa zehaztu daiteke!}$$

$\{\hat{H}, \hat{T}_{12}, \hat{S}_z, \hat{S}^2\}$-k BTMB osatzen dute, kasu honetan $[\hat{H}, \hat{S}^2] = [\hat{H}, S_z] = 0$ delako ⇒ ψ-ren aldibereko autofuntzioen oinarria kalkulatu daiteke.

## BI FERMIOIEN (s=1/2) SPIN OSOAREN AUTOFUNTZIOAK:

* $\{\hat{S}_{1z}, \hat{S}_{2z}\} \rightarrow \{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\}$ ($m_{s_1}$, $m_{s_2}$)

* Spin osoaren autofuntzioen oinarria eraikiko dugu: $S^2 = (\vec{S}_1 + \vec{S}_2)^2$, $S_z$

$S^2$-ren autobalioak → $s(s+1)\hbar^2$ , $s \in \mathbb{N}$

$S_z$-ren autobalioak → $m_s\hbar$ , $m_s \in \mathbb{Z}$

* Baina zeintzuk dira $s$ eta $m_s$-ren balio posibleak?

$$m_s = m_{s_1} + m_{s_2} \quad \text{eta} \quad s \in (|s_1 - s_2|, s_1 + s_2)$$

Kasu honetan beraz ⇒ $m_s = -1, 0, 1$ eta $s = 0, 1$

* Oinarria : $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle\}$ ($s$, $m_s$) 4 egoera (lehen bezala)

Egoera hauek golko oinarrien adierazi ditugu (Clebsch-Gordan)

$s, m_s$ maximotik hasiko gara: (gero $S_-$-z baliatu baten jarteko)

Tripletea

* $S=1$ eta $m_S=1$ → $m_S = m_{S_1} + m_{S_2}$ dugunez aukera bakarra $m_{S_1} = m_{S_2} = \frac{1}{2}$ izatea da ⇒ $|1,1\rangle = |++\rangle$

* $S=1$ eta $m_S=0$ → $S_-|S,m_S\rangle = \hbar\sqrt{S(S+1)-m_S(m_S-1)}\,|S,m_S-1\rangle$ dugunez eta $S_- = S_{1-} + S_{2-}$ denez honrela izango da $|1,0\rangle$:

$$|1,0\rangle = \frac{1}{\hbar\sqrt{2}} S_-|1,1\rangle = \frac{1}{\hbar\sqrt{2}}(S_{1-}+S_{2-})|++\rangle = \frac{1}{\hbar\sqrt{2}}(\hbar|-+\rangle + \hbar|+,-\rangle) =$$

$$\frac{1}{\sqrt{2}}(|-,+\rangle + |+,-\rangle)$$ Tripleteak ($m_S = -1,0,1$)

* $S=1$ eta $m_S=-1$ → Prozedura bera aplikatuz edo kontrakoz aukera bakarra $m_{S_1} = m_{S_2} = -1/2$ izatea dela ⇒ $|1,-1\rangle = |--\rangle$

* $S=0$ eta $m_S=0$ → Azpiespazioz aldatu gara, ezin gara aurrekoetan oinarritu ($S_-$-ek $m_S$-ren balioa baino ez du aldatzen) → Kontrakoa behar gara $m_S=0$ izenda $m_{S_1} = 1/2$ eta $m_{S_2} = -1/2$ edo $m_{S_1} = -1/2$ eta $m_{S_2} = 1/2$ izan daitezkeela eta aurrekoen perpendikularra izan behar dela.

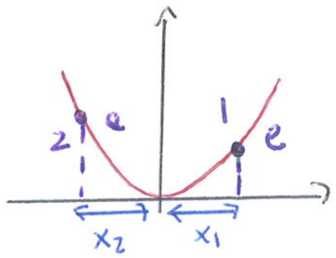
$$|0,0\rangle = \alpha|+,-\rangle + \beta|-,+\rangle \Rightarrow \langle 1 0|0 0\rangle = \frac{1}{\sqrt{2}}(\alpha+\beta) = 0 \rightarrow \alpha = -\beta$$

Normalizazioa kontuan hartuz ⇒ $|0,0\rangle = \frac{1}{\sqrt{2}}(|-,+\rangle - |+,-\rangle)$ Singleteа

*! Spin osoa truke eragilearekin trukakorra denez spin osoaren autofuntzioak bai simetrikoak edo antisimetrikoak (bi partikulen esangarriak trukatzerakoan) izango dira. $S=1$ denean autofuntzioak simetrikoak dira eta $S=0$ denean antisimetrikoak (simetria txandakatuz doa → $S$ handienekoa azpiespazioan simetrikoak, gero antisimetrikoak...)

# PARTIKULA INDEPENDENTE eta BEREIZTEZINAK ($s=1/2$): SPIN OSOA ZEHAZTURIK DUTEN EGOERA GELDIKORRAK

- Bi partikula independente eta bereiztezinak ditugu, $s=1/2$ izanik, potentzial harmoniko simple batean. Spin osoa zehaztuta duten hamiltondarren autofuntzioak aztertuko ditugu.




$$\hat{H} = \underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx_1^2} + \frac{1}{2}Kx_1^2}_{\hat{H}_1} \underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + \frac{1}{2}Kx_2^2}_{\hat{H}_2} \qquad \left(\omega = \sqrt{\frac{K}{m}}\right)$$

- Hamiltondarrean aldagai espazialak baino ez direnez agertzen, egoerak banandu ahal izango ditugu; alde batetik zati espaziala eta bestetik spinarekin lotutakoa:

$$|\psi\rangle = |\psi\rangle_{espaz.} \otimes |\psi\rangle_{spina} \quad , \quad E_{n_1 n_2} = (1+n_1+n_2)\hbar\omega$$

- Elektroiak izanik egoera antisimetrikoa izan behako da eta spin osoaren autofuntzioak antisimetrikoak edo simetrikoak (trukatze eragilearekin trukatzera delako)

- Oinarrizko egoera:

$n=2$ ——— $\frac{5\hbar\omega}{2}$

$n=1$ ——— $\frac{3\hbar\omega}{2}$

$n=0$ —↑↓— $\frac{\hbar\omega}{2}$

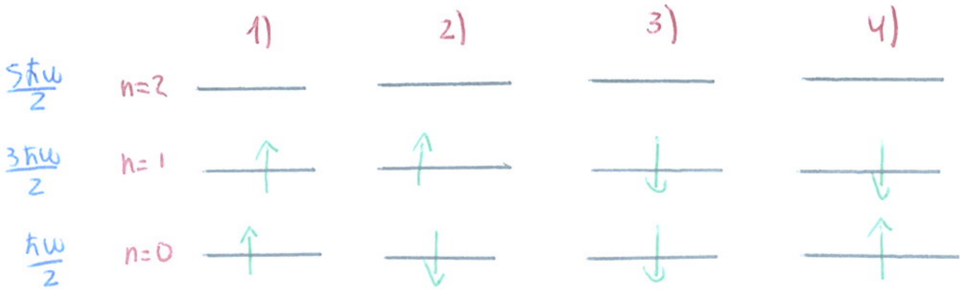
$$\Longrightarrow \quad E_0 = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{2} = \hbar\omega$$

$$g_0 = 1$$

$$|\psi_0\rangle = \underbrace{|0\rangle_1 |0\rangle_2}_{espaziala} \otimes \underbrace{|\overset{s}{0}, \overset{m_s}{0}\rangle}_{spin}$$

↳ Hau simetrikoa denez spinari dagokion zatia derrigorrez antisimetrikoa izan behako da $\Longrightarrow S=0 \leftrightarrow |0,0\rangle$

- Lehenengo egoera kitzikatua: $E_1 = \frac{\hbar\omega}{2} + 3\frac{\hbar\omega}{2} = 2\hbar\omega \rightarrow g_1 = 4$




Elkarpen espaziala: ↗ Simetrikoa

$\frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2+|1\rangle_1|0\rangle_2\}$ edo

$\frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2-|1\rangle_1|0\rangle_2\}$ ↘ Antisimetrikoa

Elkarpen espaziala simetrikoa bada → spinari dagokiona antisimetrikoa:

* $|\Psi_1^a\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2+|1\rangle_1|0\rangle_2\}\otimes|\overset{s}{0},\overset{m_s}{0}\rangle$

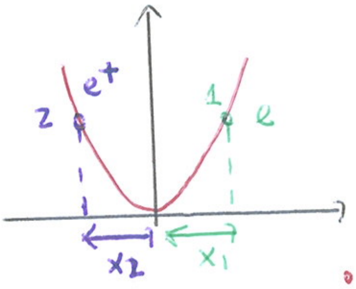
Elkarpen espaziala antisimetrikoa bada → spinari dagokiona simetrikoa:

* $|\Psi_1^b\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2-|1\rangle_1|0\rangle_2\}\otimes|\overset{s}{1},\overset{m_s}{-1}\rangle$
* $|\Psi_1^c\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2-|1\rangle_1|0\rangle_2\}\otimes|\overset{s}{1},\overset{m_s}{0}\rangle$
* $|\Psi_1^d\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2-|1\rangle_1|0\rangle_2\}\otimes|\overset{s}{1},\overset{m_s}{1}\rangle$

- Aurrekoekin alderatuz, orain eraikitako uhin-funtzioek spin osoa zehaztua dute ⇒ abantaila.
- Gainera, egoera hauek endekatuak direnez hauekin eraikitako edozein konbinazio lineal hamiltondarraren autofuntzioa izango da ere, baina beste ezaugarri batzuk izango dituzte. (ez dute spin osoa zehaztua izango...)

## PARTIKULA INDEPENDENTE eta BEREIZGARRIAK ($s=1/2$): SPIN OSOA ZEHAZTURIK DUTEN EGOERA GELDIKORRAK.

Demagun bi partikula independente eta bereizgarri ditugula, spina $s=1/2$ izanik, dimentsio bateko osziladore harmonikoan. Kasu honetan ere, spin egoera zehaztuta duten Hamiltondarraren autofuntzioak aztertuko ditugu:

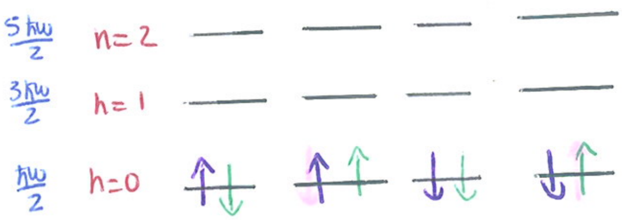



$$H = \underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx_1^2} + \frac{1}{2}Kx_1^2}_{\hat{H}_1} \underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx_2^2} + \frac{1}{2}Kx_2^2}_{\hat{H}_2}$$

- Hamiltondarean spinaren aldagaiirik ez dagoenez, egoerak banandu ahal izango ditugu; alde batetik espaziala eta bestetik spinarena:

$$|\psi\rangle = |\psi\rangle_{espaz.} \otimes |\psi\rangle_{spina}, \quad E_{n_1,n_2} = (1+n_1 n_2)\hbar\omega, \quad \omega = \sqrt{\frac{K}{m}}$$

- Oinarrizko egoera:




- $E_0 = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{2} = \hbar\omega$ eta $g_0 = 4$
- Partikulak bereizgarriak izanda $\Rightarrow$ uhin-funtzioa ez da antisimetrikoa / simetrikoa izan behar.

Hala ere, aukeratzeko askatasuna dugunez eta $m_1 = m_2$ izanik $[\hat{H}, \hat{P}_{12}] = 0$ denez, autofuntzio simetriko edo antisimetrikoak aukeratu ahal izango ditugu. Biak oinarrizko mailan egonda zati espaziala beti izango da simetrikoa.

$|\psi_0^a\rangle = |0\rangle_1 |0\rangle_2 \otimes |0,0\rangle \Rightarrow$ Antisimetrikoa

$$\left.\begin{aligned} |\psi_0^b\rangle &= |0\rangle_1|0\rangle_2 \otimes |1,-1\rangle \\ |\psi_0^c\rangle &= |0\rangle_1|0\rangle_2 \otimes |1,0\rangle \\ |\psi_0^d\rangle &= |0\rangle_1|0\rangle_2 \otimes |1,1\rangle \end{aligned}\right\} \text{Simetrikoak}$$

- Lehenengo maila-kitzikatua: • $E_1 = 2\hbar\omega$ eta $g_1 = 8$
- Egoera espazialari dagokionez $n_1 = 1$, $n_2 = 0$ edo $n_1 = 0$, $n_2 = 1$ izanda egoera simetriko edo antisimetrikoa izan dezakegu

Simetrikoa $\Rightarrow \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2\}$ Antisimetrikoa $\Rightarrow \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\}$

- Spinari dagokionez $|0,0\rangle$, $|1,0\rangle$, $|1,-1\rangle$ edo $|1,1\rangle$ izango dugu eta hauek egoera espazialekin nola konbinatzen ditugun arabera egoera simetriko edo antisimetrikoak lortuko ditugu:

$$|\Psi_0^1\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2\} \otimes |0,0\rangle \Rightarrow \text{Antisimetrikoa}$$

$$\left.\begin{array}{l} |\Psi_0^2\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2\} \otimes |1,-1\rangle \\ |\Psi_0^3\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2\} \otimes |1,0\rangle \\ |\Psi_0^4\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2\} \otimes |1,1\rangle \end{array}\right\} \Rightarrow \text{Simetrikoa}$$

$$|\Psi_0^5\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\} \otimes |0,0\rangle \Rightarrow \text{Simetrikoa}$$

$$\left.\begin{array}{l} |\Psi_0^6\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\} \otimes |1,-1\rangle \\ |\Psi_0^7\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\} \otimes |1,0\rangle \\ |\Psi_0^8\rangle = \frac{1}{\sqrt{2}}\{|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2\} \otimes |1,1\rangle \end{array}\right\} \Rightarrow \text{Antisimetrikoa}$$

Partikulak bereizgarriak izanda ere, posiblea dugu spin osoaren autofuntzioak eraikitzea. Gainera, bi masak berdinak izanik $[H, \hat{P}_{12}] = 0$ dugu eta egoera simetriko / antisimetrikoak eraiki daitezke.

## HIRU PARTIKULA INDEPENDENTE eta BEREIZTEZINEN edo GEHIAGOREN UHIN-FUNTZIOAK:

Sistema osoaren hamiltondarra $\Rightarrow$ partikulak independenteak izanik batezgarria da:

$$\hat{H} = \sum_i \hat{H}_0(\hat{\vec{r}}_i, \hat{\vec{p}}_i) \qquad \left(\hat{H}_0(\hat{\vec{r}}_i, \hat{\vec{p}}_i) = \frac{\hat{\vec{p}}_i^{\,2}}{2m} + V(\hat{\vec{r}}_i)\right)$$

Partikulak bereiztezinak izanda guztien hamiltondarra berdina da, $\hat{H}_b$:

- $\hat{H}_b$-ren i partikula bakarraren hamiltondarraren, autofuntzioak eta autobalioak:

$$\hat{H}_b \psi_K = \varepsilon_K \psi_K$$

- Autofuntzio hauekin sistema osoaren hamiltondarraren autofuntzioak lor ditzakegu:

$$\psi_a(1)\psi_b(2)\psi_c(3)$$ (3 partikula baditugu)

↘ 1. partikula

a, b, c ⇒ zenbaki kuantikoak (guztiak → espazialak, spinak ...)

1, 2, 3 ⇒ 1, 2 eta 3 partikulen idanko aldagaiak ($1 \to (\vec{r}_1, \vec{s}_1)$, ...)

Egoera hari dagokion autobalioa ⇒ $\varepsilon_a + \varepsilon_b + \varepsilon_c$

- Partikulak bereiztezinak direnez egoera antisimetrikoa izan behar da ⇒ (fermioiak)

antisimetrizatu egoera:

* 1 eta 2 partikulen antisimetrizatuz ⇒ $\frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) - \psi_b(1)\psi_a(2)\right]\psi_c(3)$

* 3 partikulen edozein permutazioarekin izan behar da antisimetrikoa ⇒ permutazio guztiak hartu behar dira kontuan ($3! = 6$):

Bi partikula soilik − ikurra

$\psi_a(1)\psi_b(2)\psi_c(3)$, $\psi_b(1)\psi_a(2)\psi_c(3)$, $\psi_c(1)\psi_b(2)\psi_a(3)$,

$\psi_a(1)\psi_c(2)\psi_b(3)$, $\psi_b(1)\psi_c(2)\psi_a(3)$, $\psi_c(1)\psi_a(2)\psi_b(3)$

Binaketa bi permutazio, + ikurra ($- \cdot - = +$)

Beraz ⇒ $$\psi = \frac{1}{\sqrt{6}}\{\psi_a(1)\psi_b(2)\psi_c(3) - \psi_b(1)\psi_a(2)\psi_c(3) - \psi_c(1)\psi_b(2)\psi_a(3) - \psi_a(1)\psi_c(2)\psi_b(3) + \psi_b(1)\psi_c(2)\psi_a(3) + \psi_c(1)\psi_a(2)\psi_b(3)\} = \psi_{abc}(1,2,3)$$

- Kalkulu hau errazteko ⇒ Slater-en determinantea:

gauza bera.

$$\psi_{abc}(1,2,3) = \frac{1}{\sqrt{3!}} \underbrace{\begin{vmatrix} \psi_a(1) & \psi_a(2) & \psi_a(3) \\ \psi_b(1) & \psi_b(2) & \psi_b(3) \\ \psi_c(1) & \psi_c(2) & \psi_c(3) \end{vmatrix}}_{\text{SLATER-EN DETERMINANTEA}}$$

Hauek antisimetrizatu dugulako (fermioiak kontsideratu ditugu) baina bosoiak badira simetrizatu behar da ⇒ ikur guztiak positiboak. (Slater erabili dezakegu eta gero ikur guztietan + jarri)

## HIRU FERMIOIEN ($s=1/2$) SPIN OSOAREN AUTOFUNTZIOAK:

- Demagun 3 fermioi ditugula, $s=1/2$ izanik. Hiruren spinen "baturen" balio posibleak: $s = s_1 \oplus s_2 \oplus s_3 = \underbrace{\frac{1}{2} \oplus \frac{1}{2}}_{1,0} \oplus \frac{1}{2} = \begin{cases} 1 \oplus \frac{1}{2} & \begin{cases} 3/2 \\ 1/2 \ (A) \end{cases} \\ 0 \oplus \frac{1}{2} & = 1/2 \ (B) \end{cases}$

| s | $m_s$ |
| --- | --- |
| 3/2 | −3/2, −1/2, 1/2, 3/2 |
| 1/2 (A) | −1/2, 1/2 |
| 1/2 (B) | −1/2, 1/2 |

} Guztira 8 egoera ⇒ autofuntzio hauek aztertu.

- $\{|s, m_s\rangle\}$ oinarria $\{|\pm \pm \pm\rangle\}$ oinarrian adierazi nahi dugu: (→ 8 bektore)

$S_{max} = 3/2$ :

* $m_{s\,max} = 3/2 \Rightarrow |3/2, 3/2\rangle = |+++\rangle$ (Aukera bakarra dago)

* $m_s = 1/2 \Rightarrow |3/2, 1/2\rangle = \frac{1}{\sqrt{3}\hbar} S_- |3/2, 3/2\rangle$

→ $m_s = m_{s_1} + m_{s_2} + m_{s_3} = 1/2$ izan behar delako

$|3/2, 1/2\rangle = \frac{1}{\sqrt{3}} (|++-\rangle + |+-+\rangle + |-++\rangle)$

→ aurrekoan + dagoen tokian − jarri eta alderantziz

* $m_s = -1/2 \Rightarrow |3/2, -1/2\rangle = \frac{1}{\sqrt{3}} (|--+\rangle + |-+-\rangle + |+--\rangle)$

* $m_s = -3/2 \Rightarrow |3/2, -3/2\rangle = |---\rangle$ (Aukera bakarra)

Autofuntzio guztiak simetrikoak edozein bi partikula trukatzerakoan!

$S_{min} = 1/2$ (bi modutan → $0 + \frac{1}{2}$ edo $|1 - 1/2|$ egoera ⇒ A eta B)

∗ $B \Rightarrow 0 \oplus \frac{1}{2}$ , badakigu $0 \Rightarrow \frac{1}{\sqrt{2}}\{|+-\rangle - |-+\rangle)$ Antisimetrikoa

$S=1/2$-rekin batu $\Rightarrow \frac{1}{\sqrt{2}}\{|+-\rangle - |-+\rangle\} \oplus |\pm\rangle$

$S=3/2$-ko egoerekin ortonormalak izateko

$|1/2,1/2\rangle_B = \frac{1}{\sqrt{2}}(|+-+\rangle - |-++\rangle)$ +

$|1/2,-1/2\rangle_B = \frac{1}{\sqrt{2}}(|+--\rangle - |-+-\rangle)$ -

Ez dira simetrikoak ezta erabat antisimetrikoak → bi egoera ($|\pm\rangle$) bezalako ez dituzkelako eta 3 partikula (Pauliren esklusio printzipioa) ∗

Lehenengo 2 partikulak trukatzerakoan badarik antisimetrikoak. ⇒ 3 partikula dituzkelako.

∗ Bestela, guztiz antisimetrizatzerakoan, bi spinen partikulak berdinak direnez, nulua izango litzateke.

∗ $A \Rightarrow 1 \oplus \frac{1}{2}$ → S osoa 1/2 izan behar denez eta $m_J = m_{S_1} + m_{S_2} + m_{S_3}$ dela kontuan hartuz:

$|1/2,1/2\rangle_A = a|+-+\rangle + b|-++\rangle + c|-++\rangle$ izango da ⇒ ortogonaltasuna

$|3/2,1/2\rangle$ eta $|1/2,1/2\rangle_B$-rekin aplikatuz + normalizatuz:

$$|1/2,1/2\rangle_A = \frac{1}{\sqrt{6}}\{2|++-\rangle - |+-+\rangle - |-++\rangle)$$

Lehenengo 2 partikulen spinak trukatzerakoan simetrikoa da.

$$|1/2,-1/2\rangle_A = \frac{1}{\sqrt{6}}(2|--+\rangle - |-+-\rangle - |+--\rangle)$$

Lehenengo 2 partikulen spinak trukatzerakoan simetrikoa da.

Baina nola lortu spin egoera guztiz antisimetrikoa? Zati espaziala kontuan hartu.

- Ad: $S=3/2$ → simetrikoa denez zati espaziala antisimetrikoa hartu.

$$\Psi(1,2,3) = \underbrace{\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)}_{\text{Antism.}}\ \underbrace{\chi_{S=3/2,m_S}}_{\text{sim.}}$$

- $S=1/2$ → Beraketa hau ezin da egin, erabat antisimetrikoa / simetrikoa den egoerak ez dituzkelako.

* $|1/2, 1/2\rangle_B$ adibidez $\Rightarrow |1/2, 1/2\rangle_B = \frac{1}{\sqrt{2}}\left(|+-+\rangle - |-++\rangle\right)$

1 eta 2 partikuletan spinari dagokion zatia antisimetrikoa denez, $\psi(\vec r_1, \vec r_2, \vec r_3)$ zati espazialarekin biderkatzean, 1 eta 2 partikulekin simetrikoa den bat aukeratu.

$$\Psi(1,2,3) = \underbrace{\psi(\vec r_1, \vec r_2}_{\text{Sim}}, \vec r_3)\, \underbrace{\chi(s_1, s_2}_{\text{Antisim}}, s_3)$$

Baina guk edozein bi partikulekin antisimetrikoa izatea nahi dugu $\Rightarrow$ bi termino gehitu:

$$\Psi(1,2,3) = \psi(\vec r_1, \vec r_2, \vec r_3)\, \chi_B(s_1, s_2, s_3) + \psi(\vec r_2, \vec r_3, \vec r_1)\, \chi_B(s_2, s_3, s_1) + \psi(\vec r_3, \vec r_1, \vec r_2)\, \chi_B(s_3, s_1, s_2)$$

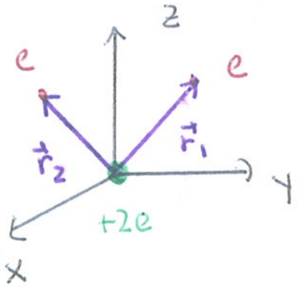
Erabat antisimetrikoa eta S-ren autofuntzioa

- Nahiz eta S neurmenekin autofuntzioak simetrikoak izan, S txikiagoko azpiespazioetan autofuntzioak ez dira erabat simetriko edo antisimetrikoak (askoz zailagoa antisimetrizatzea / simetrizatzea $\Rightarrow$ oso astuna)

## HELIO ATOMOAREN SARRERA:

→ H atomoa eta gero daukagun atomorik sinpleena

Hidrogeno atomoaren ostean daukagun atomoa Helio atomoa da: $2e^-$ eta $2p^+$.
(Isotopoa bada neutroiak ere)




3 partikula dituguela kontsideratu dezakegu $\Rightarrow$ klasikoki 3 gorputzen problema ezin da analitikoki ebatzi → kuantikoki ezin izango dugu ere zehaztasunez ebatzi

Sistema deskribatzeko dugun hamiltondarra:

→ zailtasuna hemen

$$\hat H = \underbrace{-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{r_1}}_{\hat H_1} \underbrace{- \frac{\hbar^2}{2m}\nabla_2^2 - \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{r_2}}_{\hat H_2} + \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\vec r_1 - \vec r_2|}}_{V_{12}}$$

1. partikularen energia zinetikoa

1. partikularen eta nukleoaren arteko energia potentziala

$V_{12}$ → bi elektroien arteko elkarrekintza

Azkenengo elkarpenak zailtzen ditu kalkuluak ⇒ zenbait hurbilketa egingo ditugu. Adibidez, arbuiatzen badugu bi elektroi independente izango ditugu eta emaitza zehatza lortuko dugu.

## HELIO ATOMOAREN OINARRIZKO EGOERA (PERTURBAZIO-TEORIA):

$$\hat{H} = \underbrace{\underbrace{-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_1}}_{\hat{H}_1} \underbrace{- \frac{\hbar^2}{2m}\nabla_2^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_2}}_{\hat{H}_2}}_{\hat{H}_0} + \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\vec{r}_1 - \vec{r}_2|}}_{\hat{H}_{elek.}}$$

He-ren kasuan $Z=2$

* Perturbatiboki aztertuko dugu lehen ordenako perturbazioen teoria aplikatuz:

- $\hat{H}_0$ ⇒ ore faktorizatuko hamiltondarra → bi elektroiak independenteki aztertu: autobalio eta autofuntzioak ezagunak dira
- $\hat{H}_{elk}$ ⇒ perturbazio elkarrekintza

* Ho-ren autobektoreak: $\hat{H}_0^i \underbrace{|n_i\, l_i\, m_{l_i}\, m_{s_i}\rangle}_{} = \varepsilon_{n_i}^0 |n_i\, l_i\, m_{l_i}\, m_{s_i}\rangle$ →

$$\varepsilon_{n_i}^0 = \frac{-Z^2 e^4 m}{2(4\pi\varepsilon_0)^2\hbar}\cdot\frac{1}{n_i^2} = \frac{-13'6 Z^2}{n_i^2}\,(eV) \qquad \underbrace{R_{n_i l_i}(r)\, Y_{l_i}^{m_{l_i}}(\theta,\varphi)}_{\text{espaziala}}\ \underbrace{\chi_{s_i, m_{s_i}}}_{\text{spina}}$$

↗ Spinaren ekarpena

* Oinarrizko egoeran bi elektroiak $n_1 = n_2 = 1$ mailan ⇒ spin egoera ezberdina.

$$E_0 = 2\varepsilon_1^0 = -108'8\,eV \quad , \quad |\Psi_0\rangle = \underbrace{|100\rangle_1 |100\rangle_2}_{\text{Antisimetrikoa}} \otimes |\chi_{\underset{s}{0},\underset{m_s}{0}}\rangle$$

→ singletea → antisimetrikoa izateko

Egoera hau ⇒ parahelioa

* Elkarrekintza ⇒ lehenengo ordenako perturbazioen teoria:

$$\underset{\nearrow \text{ lehen ordenako p. teoria}}{E_0^1} = E_0 + \Delta E_0 \quad , \quad \underset{\nearrow \text{ perturbazioa}}{\Delta E_0} = \langle\Psi_0|\hat{H}_{elk}|\Psi_0\rangle$$

- $\langle\psi_0|\hat{H}_{elk}|\psi_0\rangle = \lambda\,\langle\psi_0|\tilde{W}|\psi_0\rangle = \dfrac{e^2}{4\pi\varepsilon_0}\displaystyle\int d^3\vec{r}_1\, d^3\vec{r}_2\, \dfrac{|\psi_{100}(\vec{r}_1)|^2\,|\psi_{100}(\vec{r}_2)|^2}{|\vec{r}_1-\vec{r}_2|} =$

Spin egoera integratzean 1 lortu → $\hat{H}_{elk}$ spinaren independentea

$-\frac{5}{4} Z E_1^H$ → Hidrogeno atomoaren energia $n=1$ mailan, $-13'6$ eV

Beraz ⇒ $E_1^0 = E_0 + \Delta E_0 = -108'8\,eV + 34\,eV = -74'8\,eV$

% 31-a da perturbazioaren gehikuntza ⇒ 0 ordenako hurbilketa ez da egokia

* Emaitza esperimentala ⇒ $E_{exp} \sim -78'98$ eV (nahiko hurbil)

## BARIAZIO METODOA:

Demagun ez dugula ezagutzen Hamiltondarraren oinarrizko egoera eta horren autobalioa ⇒ bariazio metodoa erabili oinarrizko egoera horren hurbilketa lortzeko

→ Hamiltondarraren batez bestekoa oinarri horretan

→ oinarrizko egoeraren autobalioa

- $|\psi\rangle \rightarrow \langle\psi|\hat{H}|\psi\rangle = \langle\hat{H}\rangle_\psi \geq \varepsilon_0$ BETI! *[1]

- $\hat{H}|\psi_0\rangle = \varepsilon_0|\psi_0\rangle$ (ezezagunak $\varepsilon_0$ eta $|\psi_0\rangle$)

*[1] $\hat{H}|\psi_i\rangle = \varepsilon_i|\psi_i\rangle$ → $\hat{H}$-ren autobektore eta autobalioak →

edozein $|\psi\rangle$ egoera autobektore horien oinarrian garatu:

$$|\psi\rangle = \sum_i c_i|\psi_i\rangle \Rightarrow \langle\psi|\hat{H}|\psi\rangle = \sum_i c_i^*\langle\psi_i|\hat{H}|\Big(\sum_j c_j|\psi_j\rangle\Big) =$$

$$\sum_i\sum_j c_i^* c_j \langle\psi_i|\hat{H}|\psi_j\rangle = \sum_i\sum_j c_i^* c_j\,\varepsilon_j\,\langle\psi_i|\psi_j\rangle = \sum_{i,j} c_i^* c_j\,\varepsilon_j\,\delta_{ij} =$$

$$\sum_i |c_i|^2\varepsilon_i = |c_0|^2\varepsilon_0 + |c_1|^2\varepsilon_1 + \dots \underbrace{\geq}_{\varepsilon_0 \leq \varepsilon_i\ \forall i} \varepsilon_0\,\underbrace{\Big(\sum_i |c_i|^2\Big)}_{\text{normalizatuta}} = \varepsilon_0$$

- Metodo bariazionala ⇒ oinarrizko egoera izan daitekeen $|\psi_\alpha\rangle$ uhin-funtzioa proposatu $\alpha$ parametroaren mende $\Rightarrow |\Psi_\alpha\rangle \Rightarrow \langle\Psi_\alpha|\hat{H}|\Psi_\alpha\rangle = E_\alpha$

kalkulatu $\Rightarrow E_\alpha \geq E_0$ izango da $\rightarrow$ minimizatu $E$ $\alpha$-rekiko

$$\frac{\partial E_\alpha}{\partial \alpha} = 0 \quad \text{(minimoa)}$$

Zailtasuna $\Rightarrow |\Psi_\alpha\rangle$ proposatzea $\rightarrow$ proposamena egokia ez bada $E_\alpha$ (nahiz eta minimizatu) $E_0$-tik oso urrun egon daiteke.

## HELIO ATOMOAREN OINARRIZKO EGOERA BARIAZIO-METODOA APLIKATUZ:

$$\hat{H} = \underbrace{-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{2e^2}{4\pi\varepsilon_0}\frac{1}{r_1}}_{\hat{H}_1} \underbrace{-\frac{\hbar^2}{2m}\nabla_2^2 - \frac{2e^2}{4\pi\varepsilon_0}\frac{1}{r_2}}_{\hat{H}_2} + \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\vec{r}_1 - \vec{r}_2|}}_{\hat{H}_{elk.}}$$

- Metodo bariazionala aplikatuko dugu $\hat{H}$-ren oinarrizko egoera kalkulatzeko $\rightarrow$ $\hat{H}$ bere osotasunean kontsideratuko dugu eta ez $\hat{H}_0 + \hat{H}_{elk.}$ perturbazio teorian bezala.

- Lehen urratsa $\Rightarrow$ proposamena:


2. elektroia $\Rightarrow$ ikusten duen nukleoaren karga ez da $2e$ $\rightarrow$ pantailatze efektua $\rightarrow$ karga efektiboa espero dugu $\rightarrow$ nukleoaren karga $2e$ baino txikiagoa

1. elektroiak osoaren duen karga dentsitatea puntu ezberdinetan banatuta dagoelako

(Argumentu bera 1 eta 2 elektroiak trukatuz)

* Ideia hau kontuan hartuz $\Rightarrow$ elektroien uhin funtzioak atomo hidrogenoideena bana $Z$ zenbaki atomikoarekin, ez $Z$: $\Psi_Z = R_{10}(r_1)\,Y_0^0(\theta_1,\varphi_1)\,R_{10}(r_2)\,Y_0^0(\theta_2,\varphi_2)\,\chi_{00}^{\,S}{}_{\,m_S}$

$R_{10} = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr_1/a_0}$ ; $Y_0^0 = \frac{1}{\sqrt{4\pi}}$

Spin egoera antisimetrikoa $\Psi_Z$ antisimetrikoa izateko (singleta)

- Bigarren urratsa $\Rightarrow \langle\Psi_Z|\hat{H}|\Psi_Z\rangle$ kalkulatu.

$$\langle\Psi_Z|\hat{H}|\Psi_Z\rangle = \langle\Psi_Z|\hat{H}_1|\Psi_Z\rangle + \langle\Psi_Z|\hat{H}_2|\Psi_Z\rangle + \langle\Psi_Z|\hat{H}_{elk}|\Psi_Z\rangle = \langle\Psi_Z|\hat{T}_1|\Psi_Z\rangle +$$

$$\langle \Psi_Z | \hat{V}_1 | \Psi_Z \rangle + \langle \Psi_Z | \hat{T}_2 | \Psi_Z \rangle + \langle \Psi_Z | \hat{V}_2 | \Psi_Z \rangle + \langle \Psi_Z | \hat{H}_{elek} | \Psi_Z \rangle$$

* Kalkuluak errazteko ⇒ Virialaren teorema: $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{r}$ bada

$\Psi_Z = R_{10} Y_0^0$ da oinarrizko egoerari dagokion uhin-funtzioa $\Rightarrow \langle \Psi_Z | \hat{H} | \Psi_Z \rangle =$

$$-\frac{Z^2 e^4 m}{2(4\pi\varepsilon_0)^2\hbar^2} = Z^2 E_0^H$$ hidrogeno atomoaren oinarrizko egoera

$$\frac{e^4 m}{2(4\pi\varepsilon_0)^2\hbar^2} = E_0^H$$

Viriala ⇒ $\langle \hat{T} \rangle_{\Psi_Z} = -2\langle \hat{V} \rangle_{\Psi_Z} \rightarrow \langle \hat{T} \rangle_{\Psi_Z} + \langle \hat{V} \rangle_{\Psi_Z} = \langle \hat{H} \rangle_{\Psi_Z}$

$$\langle \hat{T} \rangle_{\Psi_Z} = -Z^2 E_0^H \,, \quad \langle \hat{V} \rangle_{\Psi_Z} = 2Z^2 E_0^H$$

Orduan $\langle \Psi_Z | \hat{T}_1 | \Psi_Z \rangle = \langle \Psi_Z | \hat{T}_2 | \Psi_Z \rangle = -Z^2 E_0^H$ ; $\hat{V}_1$ eta $\hat{V}_2$-n ordea ezberdintasuna $Z$ beharrean 2 dugulako hamiltondarrean. → zati $Z$ bider 2 egin ⇒ $\langle \hat{V}_1 \rangle_{\Psi_Z} = \frac{2}{Z}\langle \hat{V} \rangle_{\Psi_Z} = 4Z E_0^H = \frac{2}{Z}\left\langle \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{r} \right\rangle_{\Psi_Z} = \left\langle \frac{2e^2}{4\pi\varepsilon_0}\frac{1}{r} \right\rangle_{\Psi_Z}$

* $\langle \Psi_Z | \hat{H}_{elek} | \Psi_Z \rangle$ kalkulatzeko integrala egin behar da:

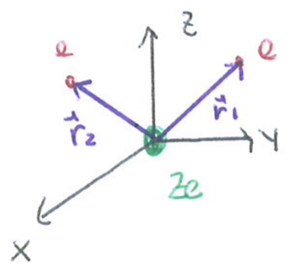
$$\langle \Psi_Z | \hat{H}_{elek} | \Psi_Z \rangle = \frac{e^2}{4\pi\varepsilon_0}\int d\vec{r}_1^{\,3} d\vec{r}_2^{\,3} \frac{|\Psi_Z(\vec{r}_1)|^2 |\Psi_Z(\vec{r}_2)|^2}{|\vec{r}_1 - \vec{r}_2|} = -\frac{5}{4} Z E_0^H$$

$\Rightarrow \langle \hat{H} \rangle_{\Psi_Z} = E_Z = \left(\frac{27}{4}Z - 2Z^2\right) E_0^H$ → metodo bariazionala aplikatu:

$\frac{dE_Z}{dZ} = E_0^H \left(\frac{27}{4} - 4Z\right) = 0 \rightarrow Z_0 = \frac{27}{16} \simeq 1'69 < 2$ (Hasierako ideiarekin bat dator) $\Rightarrow E_0 = E_{Z_0} = -77'5\ eV$ ($E_{exp} \simeq 78'98\ eV$)

% 1'9-ko errorea ⇒ hobekuntza lortu dugu.

## HELIO ATOMOAREN LEHENENGO EGOERA KITZIKATUAK (PERTURBAZIO-TEORIA)




- $\hat{H} = \hat{H}_0^1 + \hat{H}_0^2 + \hat{H}_{elk} = \hat{H}_0 + \hat{H}_{elk}$ (anelioa)

$$\hat{H}_0^i \rightarrow \hat{H}_0^i \underbrace{|n_i\, l_i\, m_{l_i}\, m_{s_i}\rangle}_{R_{n_i l_i}(r_i)\, Y_{l_i}^{m_{l_i}}(\theta_i, \varphi_i)\, \chi_{s_i m_{s_i}}} = \varepsilon_{n_i}^0 |n_i\, l_i\, m_{l_i}\, m_{s_i}\rangle$$

$$\varepsilon_{n_i}^0 = -\frac{Z^2 e^4 m}{2(4\pi\varepsilon_0)^2\hbar^2} \cdot \frac{1}{n_i^2} = -\frac{13'6}{n_i^2} Z^2 \ (eV)$$

- Lehenengo egoera kitzikatua:

$$\begin{cases} n=2,\ \varepsilon_2 = -13'6\ eV \rightarrow l=1,0\ ;\ m_l = \pm 1, 0 \\ n=1,\ \varepsilon_1 = -54'4\ eV \rightarrow l=0,\ m=0 \end{cases}$$

$E_1 = \varepsilon_1 + \varepsilon_2 = -68\ eV$

(endakatua) $\rightarrow$ $|\psi_p\rangle = \frac{1}{\sqrt{2}} (|100\rangle_1 |2\, l\, m_l\rangle + |2\, l\, m_l\rangle |100\rangle_2) \otimes |\chi_{0,0}\rangle$ ($s$, $m_s$)

parahelioa

egoera espaziala simetrikoa $\rightarrow$ spin egoera antisimetrikoa ordua $\rightarrow$ singletea

$\rightarrow$ ortohelioa

$\rightarrow$ egoera espaziala antisimetrikoa $\rightarrow$ spin egoera simetrikoa

$$|\psi_o\rangle = \frac{1}{\sqrt{2}} (|100\rangle_1 |2\, l\, m_l\rangle - |2\, l\, m_l\rangle_1 |100\rangle_2) \otimes |\chi_{1, m_s}\rangle$$

↑ espaziala ↖ ↗ spina

Guztira 16-ko endakapena. $\Rightarrow$ $|\psi_p\rangle \rightarrow 4$ , $|\psi_o\rangle \rightarrow 4 \cdot 3$

- Elkarrekintza kontuan hartzeko $\rightarrow$ perturbazio teoria aplikatu, 1. ordenakoa.

Endakapena dugunez $W = \hat{H}_{elk}$. diagonalizatu beharko genuke 1. egoera kitzikatuari dagokion autobektoreen oinarria $\Rightarrow$ 16x16-ko matrizea denez ezin zaio aztertzean $\Rightarrow$ endakapena nola apurtzen den aztertuko dugu bakoitzik egoera orbital batean.

$$\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0} \int d^3r_1 d^3r_2 \frac{|\psi_{100}(\vec{r}_1)\psi_{2lm_l}(\vec{r}_2) \pm \psi_{2lm_l}(\vec{r}_1)\psi_{100}(\vec{r}_2)|^2}{|\vec{r}_1 - \vec{r}_2|} \quad *^1$$

($+ \rightarrow p$, $- \rightarrow o$)

$$\overbrace{E_{p,o}^1}^{\text{para / orto}} = E_1 + \Delta E_{p,o} = E_1 + \langle \psi_{p,o} | \hat{H}_{elk} | \psi_{p,o} \rangle$$

$*^1$ $|\psi_{100}(\vec{r}_1)\psi_{2lm_l}(\vec{r}_2) \pm \psi_{2lm_l}(\vec{r}_1)\psi_{100}(\vec{r}_2)|^2 = |\psi_{100}(\vec{r}_1)|^2 |\psi_{2lm_l}(\vec{r}_2)|^2 \pm \psi_{100}^*(\vec{r}_1)\psi_{2lm_l}^*(\vec{r}_2) \cdot$
$\psi_{2lm_l}(\vec{r}_1)\psi_{100}(\vec{r}_1) \pm \psi_{2lm_l}^*(\vec{r}_1)\psi_{100}^*(\vec{r}_2)\psi_{100}(\vec{r}_1)\psi_{2lm_l}(\vec{r}_2) + |\psi_{2lm_l}(\vec{r}_1)|^2 |\psi_{2lm_l}(\vec{r}_2)|^2$

integratzean elkarren kontra emaitza berdina izango da eta beste biena baita (o)

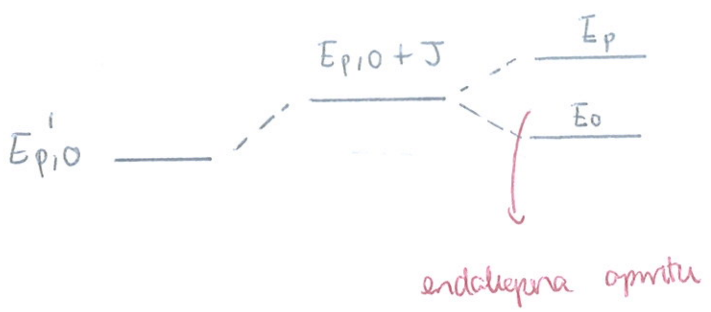
Orduan ⇒ $\Delta E_{p,o} = \dfrac{e^2}{4\pi\varepsilon_0}\displaystyle\int d^3\vec{r}_1\, d^3\vec{r}_2 \, \frac{|\psi_{100}(\vec{r}_1)|^2\, |\psi_{2lm_l}(\vec{r}_2)|^2}{|\vec{r}_1-\vec{r}_2|} \; +$ → J deitzen zaio; elkarpen zuzena → Klasikoki bi karga dentsitateen biderkadura

$\pm \dfrac{e^2}{4\pi\varepsilon_0}\displaystyle\int d^3\vec{r}_1\, d^3\vec{r}_2 \, \frac{\psi^*_{100}(\vec{r}_1)\,\psi^*_{2lm_l}(\vec{r}_2)\,\psi_{2lm_l}(\vec{r}_1)\,\psi_{100}(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|} =$ → K deitzen zaio; truke elkarpena

$$\underbrace{J_{100;2lm_l}}_{>0} \; \overset{+\to p}{\underset{-\to o}{\pm}} \; \underbrace{K_{100;2lm_l}}_{>0}$$

→ Azterketa kualitatiboa

• Zenbat du energiak baxuena? Hori izango da gure 1. egoera kitzikatua




1. egoera kitzikatua orto egoera da → $S=1$ spinarekin. (spin handiena duena)

# HELIO ATOMOAREN OINARRIZKO eta LEHENENGO EGOERA KITZIKATUEN ESKEMATXOA (NOTAZIO ESPEKTROSKOPIKOA)

• Notazio espektroskopikoa ⇒ 1 $e^-$ bakar eta honi lotzen denean spin orbita arteko akoplamendurik etab. ez badago egoerak izendatzeko zenbaki kuantiko hauek erabiltzen dira: $n$ $l$

$l$ adierazteko letra ezberdinak erabiltzen dira $\begin{cases} l=0 \to s & l=3 \to f \\ l=1 \to p & l=4 \to g \\ l=2 \to d & \text{! alfabetikoki} \end{cases}$

Izenak lerro espektraletatik: $\begin{cases} s \to \text{sharp}, & p \to \text{principal} \\ d \to \text{diffuse}, & f \to \text{fundamental} \end{cases}$

$n=2$ $l=1$ bada ⇒ $2p$.

• 2 $e^-$ edo gehiago edo 1 $e^-$ eta LS-ren akoplamendua badugu ⇒ notazio espektroskopiko ezberdina → $^{2S+1}L_J$

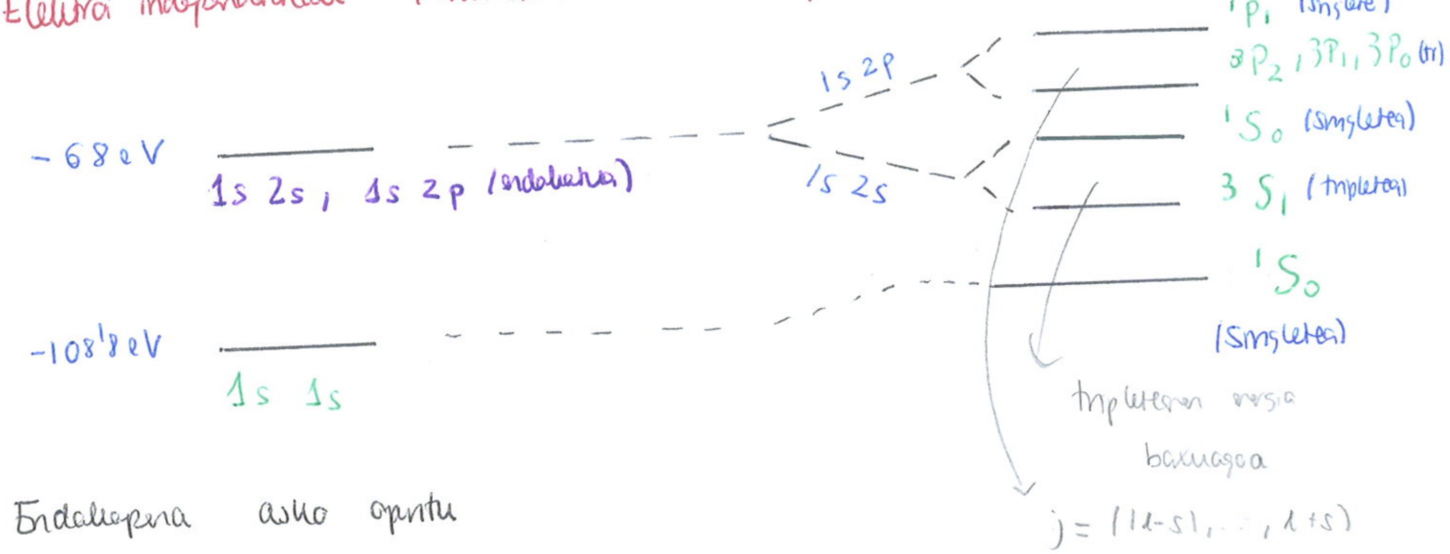
Spin osoaren lotutako zenbaki kuantikoa

momentu angeluar osoaren lotutako zenbaki kuantikoa (letra larriz)

$(\vec{S}+\vec{L})^2 = J^2$-ren zenbaki kuantikoa.

Helio atomoa:

Elektroi independenteak (elkarrekintzarik ez) ; Elkarrekintza bai.



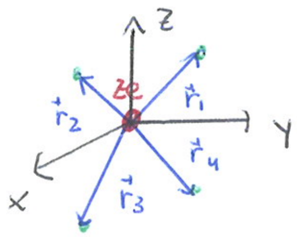

Endakapena asko apurtu

# ATOMO ELEKTROIANITZAK (LEHENGO HURBILKETA)

$$\hat{H} = \sum_{i} \left\{ -\frac{\hbar^2}{2m}\nabla_i^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_i} \right\} + \sum_{i<j} \frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\vec{r}_i-\vec{r}_j|}$$

$e^-$ kopurua

elkarrekintza birritan ez zenbatzeko




$\hat{H}$-ren autobektoreak kalkulatzeko arazoa elkarrekintzei dagokion energia potentzialean daukagu, $r_i$ eta $r_j$ nahasten direlako. ⇒ hurbilketak aplikatu

Elkarrekintza hori perturbaziotzat hartzen badugu perturbazio metodoa aplikatuko dugu.

* Zero ordenako hurbilketan arbuiatu egiten da orduan ⇒ zehatztasun osoz lor daitezke H-ren autobalio eta autobektoreak : $E_n^0 = -\frac{Z^2 e^4 m}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2}$

(partikula bakoitzarena) ⇒ energia maila ezberdinak sortuko dira.

(Notazio espektroskopikoa erabiliz)

⋮ ⋮

$3s^2$ $3p^6$ $3d^{10}$ $n = 3$

$2s^2$ $2p^6$ → 6 $\bar{e}$ sartzen dira $n = 2$

$1s^2$ → 2 $\bar{e}$ sartzen dira $n = 1$

Elektroi kopuruaren arabera elektroiak maila hauetan kokatzen joango dira, kontuan hartuz maila bakoitzean zenbat elektroi egon daitezkeen etab.

Adibidez, 3 $\bar{e}$ izango bagenitu $1s^2$ lehenengo biak egongo lirateke eta hirugarrena $2s^2$ edo $2p^6$ → hurbilketa honetan erabateko endakapena dago.
↳ printzipioz edozeinetara.

Laburpena:

| n | l | Endakapena (→ $2n^2$) | Guztira |
|---|---|---|---|
| 1 | 0 | 2 | 2 |
| 2 | 0,1 | 8 | 10 |
| 3 | 0,1,2 | 18 | 28 |
| 4 | 0,1,2,3 | 32 | 60 |
| ⋮ | ⋮ | ⋮ | ⋮ |

Elektroi kopurua 2, 10, 28, 60... etab. direnean n mailak erabat okupatuta geratzen dira → atomoa oso esferikoa eta bere ionizazio energia oso altua izango da. (oso zaila izango da ionizatzea)
↳ Atomoari eman behar zaion energia minimoa ionizatzeko

Elektroi kopurua aipatutako kopuru hauek baino 1 handiagoa bada hurrengo mailan geratuko da isolatuta elektroi bakarra ⇒ nukleoarekiko duen lotura-energia txikiagoa izango da ⇒ errazago ionizatuko da.

Esperimentalki hau ez da betetzen: ionizazio altuena dutenak

$Z=2$ (He), 10 (Ne), 18 (Ar), 36 (Kr), ...

Ahulenak, ionizazio energia txikiena dutenak, alkalinoak dira (Li, Na, K...) $\rightarrow$ azkenengo elektroiaren lotura nukleoarekin ahulena dutenak ($Z=3, 11, 19\dots$)

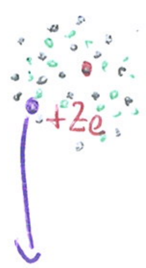
Gure suposizioaren arabera ionizazio energia, azkenengo elektroiari eman behar zaion energia $\propto Z^2$ izango da baina alkalinoen kasuan adibidez ikusi daiteke guztien ionizazio-energia oso antzekoa dela. (nahiz eta zenbaki atomiko oso ezberdina izan)

Zero ordenako perturbazioen teoria (elkarrekintza arbuiatu) ez da egokia!

## EREMU ZENTRALAREN HURBILKETA:

$$\hat{H} = \sum_i \left\{ -\frac{\hbar^2}{2m}\nabla_i^2 - \frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{r_i} \right\} + \underbrace{\sum_{i<j} \frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\vec{r}_i-\vec{r}_j|}}_{\text{elkarrekintza}}$$

Atomo elektroianitza




$e^-$ batekin lotutako dentsitate probabilitatea

Beste $e^-$ batekin lotutako dentsitate probabilitatea

($e^-$ balitzateke karga dentsitate bat)

Hemen $e^-$ bat badugu elektroi honek ez du nukleoaren karga ikusiko balebik, beste elektroiek karga hori apantailatzen dutelako $\Rightarrow$ energia potentzial efektibo bat jasango du (nukleoak + beste elektroiek eginadakoa). Hurbilketa batean elektroiak quasi-independentetzat hartuko ditugu, orduan: $\hat{H} \simeq \sum_i \hat{H}_i$

non $\hat{H}_i = -\frac{\hbar^2}{2m}\nabla_i^2 + U_i(r_i)$ → erraztuko zentraltzat hartu

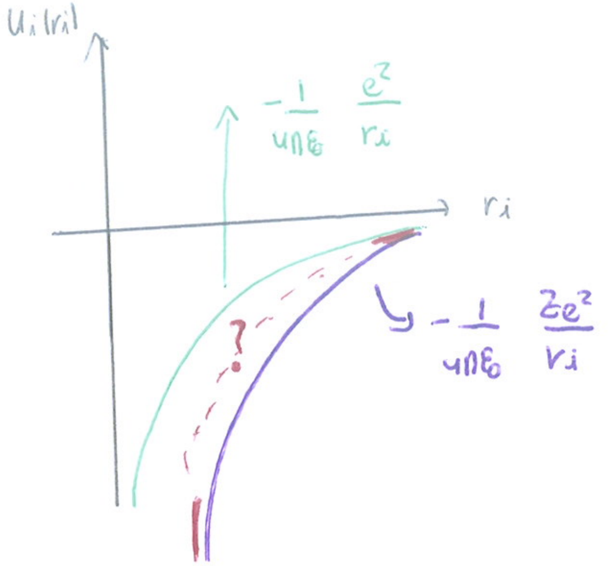
↳ energia potentzial efektiboa

(eremu potentzial zentrala)

$$U_i(r_i) = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_i} + V_{ef}(r_i) = \begin{cases} -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_i} & r_i \to 0 \\ -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_i} & r_i \to \infty \end{cases}$$

($r_i \to 0$:) oso hurbil, apantailamendua ia nulua

($r_i \to \infty$:) beste $e^-$ guztiek apantailatu $(Z-1)\,e^-$-ek




— ⟹ erreala ; erdibideko $r_i$-ren balioekin ez gertatzen den ez dakigu

## HARTREE-REN METODOA:

$$\hat{H} = \sum_i \left\{ -\frac{\hbar^2}{2m}\nabla_i - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_i} \right\} + \sum_{i<j} \frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\vec{r}_i - \vec{r}_j|} \simeq \sum_i \left\{ -\frac{\hbar^2}{2m}\nabla_i^2 + U_i(r_i) \right\}$$

Potentzial efektiboa ?

- $U_i(r_i)$ ezezaguna da ⟹ hurbildu ⟹ Hartree-ren metodoa (metodo bariazionala)

Metodo bariazionala → uhin-funtzio osoaren hurbilketa bat egin, proposatu bat.

elektroi guztien aldagaiak

$$\Psi(\{\vec{r}_i\}) = \psi_{i_1}(\vec{r}_1)\,\psi_{i_2}(\vec{r}_2)\cdots\psi_{i_n}(\vec{r}_N) \quad , \quad i_j \Rightarrow \text{Uhin-funtzioaren zbk kuantikoak}$$

↳ elektroi quasi-independenteak ⟹ elektroi bakoitzaren uhin-funtzioen biderkadura

- Hurbilketan $U_i(r_i)$ zentrala da ⟹ $\{H, L^2, L_z\}$-ren aldiberako autofuntzioak aukeratu daitezke ⟹ $i_1 \equiv (n_1, l_1, m_{l_1}, m_{s_1})$

Partikulak (elektroiak) fermioiak direnez uhin-funtzioa antisimetrikoa izan beharko da, baina $\Psi(\{\vec{r}_i\})$ ez da antisimetrikoa ⇒ Hartreeren araroaketako bat.

- Hala ere, antisimetrikoaren eragina zertxo batean barneratzen du, Pauliren esklusio printzipioa aplikatzen delako ⇒ $i_j$ zenbaki kuantikoak ezin dira berdinak izan. (elektroiak energia mailak betetzen joango dira haien zenbaki kuantikoak ezberdinak izanda)

- Metodo bariazionalean parametro baten menpe jartzen da uhin-funtzioa ⇒ Hartreeren metodoan parametroak $\Psi_{ij}(\vec{r}_j)$ uhin-funtzioak izango dira; energia minimizatzerakoan uhin-funtzio horiek bete beharko duten ekuazio bat lortuko dugu. → osoa, zehatza

- $$E[\Psi] = \langle\Psi|\hat{H}|\Psi\rangle = \sum_i \int d^3\vec{r}\,'\,\Psi_i^*(\vec{r}\,')\left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r'}\right)\Psi_i(\vec{r}\,') +$$

Energiaren funtzionala

$$\frac{e^2}{4\pi\varepsilon_0}\sum_{i<j}\int d^3\vec{r}\,'d^3\vec{r}\,''\,\Psi_i^*(\vec{r}\,')\Psi_j^*(\vec{r}\,'')\frac{1}{|\vec{r}\,'-\vec{r}\,''|}\Psi_j(\vec{r}\,'')\Psi_i(\vec{r}\,')$$

Minimizatzerakoan kontuan hartzekoa ⇒ $\langle\Psi_i|\Psi_i\rangle = 1$ (normalizatuta)

Lagrangeren biderkatzailea ⇒ minimizatzeko.

↱ Lagrangeren biderkatzaileak; konstanteak (ezezagunak)

$$\mathcal{J}[\{\Psi_i\},\varepsilon_i] = E - \sum_j \varepsilon_j[\langle\Psi_j|\Psi_j\rangle - 1]$$

$$\frac{\partial\mathcal{J}}{\partial\varepsilon_i} = \langle\Psi_i|\Psi_i\rangle - 1 = 0 \Rightarrow \langle\Psi_i|\Psi_i\rangle = 1$$

Deribatu funtzionalak: $\frac{\partial\mathcal{J}}{\partial\Psi_i} = 0$ $\left(\text{edo } \frac{\partial\mathcal{J}}{\partial\Psi_i^*}\right)$ → Uhin-funtzioek bete behar dituzten ekuazioa lortu.

↳ ez dira berdinak eta baliokideak!

$$* \quad \frac{\delta E}{\delta \psi_i^*(\vec{r})} = \int^3 d^3\vec{r}\,' \delta(\vec{r}-\vec{r}\,') \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r'}\right)\psi_i(\vec{r}\,') = \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}\right)\psi_i(\vec{r}) +$$

$\psi_i^*(\vec{r}\,')$ agertzen da eta horren deribatua $\delta(\vec{r}-\vec{r}\,')$ da

beste terminoen deribatua

Beraz $\Rightarrow$ $$\frac{\delta J}{\delta \psi_i^*} = 0 \Rightarrow \left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}\right)\psi_i(\vec{r}) + \underbrace{\frac{1}{4\pi\varepsilon_0}\sum_{i\neq j}\int d^3\vec{r}\,'|\psi_j(\vec{r}\,')|^2\frac{e^2}{|\vec{r}-\vec{r}\,'|}}_{*'}\psi_i(\vec{r}) =$$

$\varepsilon_i \psi_i(\vec{r})$

Hamiltondarraren autobalio eta autofuntzioen problemaren itxura

Hartreeren ekuazioak

*' Elektroi bakoitzak beste guztiekin duen elkarrekintzaren eragina, energia potentziala $\Rightarrow$ energia potentzial efektiboa: $V_i^{ef}(\vec{r})$ → ez da zentrala → hurbildu

## EREMU ZENTRALAREN HURBILKETAKO HARTREE-REN EKUAZIOAK GARATZEKO ESTRATEGIA NUMERIKOA:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + \underbrace{V_i^{ef}(\vec{r}) - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}}_{U_i(\vec{r})}\right]\psi_i(\vec{r}) = \varepsilon_i\psi_i(\vec{r}_i)$$

Hartree-ren ekuazioak

$$V_i^{ef}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i\neq j}\int d^3\vec{r}\,'|\psi_j(\vec{r}\,')|^2\frac{e^2}{|\vec{r}-\vec{r}\,'|}$$

( Elkarrekintza klasikoaren baliokidea )

$|\psi_j(\vec{r}\,')|^2$ ~ j elektroiaren karga dentsitatea

- $\Psi[\{\vec{r}_i\}] = \psi_{i1}(\vec{r}_1)\psi_{i2}(\vec{r}_2)\cdots\psi_{iN}(\vec{r}_N)$

$V_i^{ef}(\vec{r})$ ez da zentrala → hurbildu (zentrala bada $\{H, L^2, L_z\}$-ren aldibereko autofuntzioak lortu ditzakegu → harmoniko esferikoak + parte erradiala; dimentsio bakarreko ekuazioa ebatzi behar da)

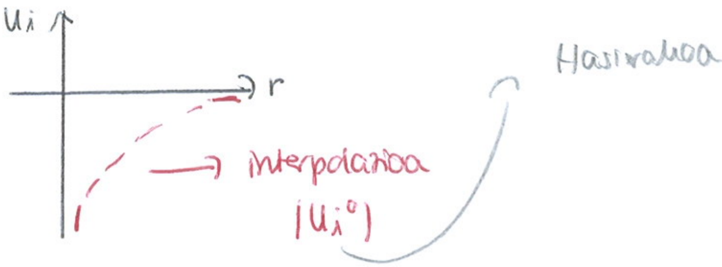
Hurbiltzeko $\Rightarrow$ angelu guztietako $V_i^{ef}(\vec{r})$-ren batez bestekoa angelu solidoa : $\sin\theta\, d\theta\, d\varphi$

• $V_i^{ef}(r) = \int d\Omega \; V_i^{ef}(\vec{r}) \cdot \frac{1}{4\pi} \Rightarrow U_i(r) = V_i^{ef}(r) - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}$

angelu solido guztien batura

Ekuazioak ebazteko ⇒ numerikoki ⇒ $U_i(r)$ ez dugu ezagutzen.

• $U_i(r)$ ez dugu ezagutzen baina limiteak bai ⇒ $U_i(r) = \begin{cases} r\to 0 & -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r} \\ r\to\infty & -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \end{cases}$




$U_i^0$-rekin Hartreeren ekuazioetan sartu eta $\psi_i^0$ lehenengo uhin-funtzioa

eta autobalioak lortu: $U_i^0 \to \psi_i^0, \varepsilon_i^0 \Rightarrow U_i^1$ berria →

$\psi_i^1, \varepsilon_i^1 \Rightarrow U_i^2$ berria → $\psi_i^2, \varepsilon_i^2 \ldots$ emaitza zehatzera eraman (ordenean)

$\psi_i = R_{nl}(r)\, Y_l^m(\theta, \varphi)$ ezezaguna → ebatzi behar duguna.

aurreko iterazioa

Ordenean zehazteko limite bat ezarri ; adibidez $\frac{|\varepsilon_i^{n-1} - \varepsilon_i^n|}{|\varepsilon_i|^n} \leq 10^{-3}$

betetzea edota autobalioentzako ⇒ emaitza zehatzetik oso hurbil

($\psi_i^n$ eta $\varepsilon_i^n$-rekin gelditu)

↳ Sistema osoaren uhin-funtzioa sortu (Pauliren esklusio printzipioa kontuan hartuz)

# HARTREE-FOCK-en METODOA:

Hartree: $\Psi[\{\vec{r}_i\}] = \psi_{i1}(\vec{r}_1)\,\psi_{i2}(\vec{r}_2)\dots\psi_{iN}(\vec{r}_N) \Rightarrow$ ez da antisimetrikoa

Hartree-Fock: $\Psi[\{\vec{r}_i\}] = \frac{1}{\sqrt{N!}} \underbrace{\begin{vmatrix} \psi_{i1}(\vec{r}_1) & \psi_{i1}(\vec{r}_2) & \cdots & \psi_{i1}(\vec{r}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{iN}(\vec{r}_1) & \psi_{iN}(\vec{r}_2) & \cdots & \psi_{iN}(\vec{r}_N) \end{vmatrix}}_{\text{Slater-en determinantea}}$ Antisimetrizatu

↳ Hasierako uhin-funtzio gisa aukeraz proposatzen da

Metodo bariazionala $\Rightarrow$ $\langle \Psi | \hat{H} | \Psi \rangle = E$ kalkulatu behako dugu.

$(\langle \psi_i | \psi_i \rangle = 1 \quad \forall i)$ Lagrangeren biderkatzaileen metodoa:

$$J = E - \sum_i \varepsilon_i \left[ \langle \psi_i | \psi_i \rangle - 1 \right] \rightarrow \text{minimizatu} \rightarrow \frac{\delta J}{\delta \psi_i(\vec{r})} = 0$$

Hartree-Fock-en ekuazioak:

$$\left[ \frac{-\hbar^2}{2m} \nabla_i^2 - \frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{1}{r} \right] \psi_i + \frac{e^2}{4\pi\varepsilon_0} \sum_{j \neq i} \int d^3\vec{r}\,' \frac{|\psi_j(\vec{r}\,')|^2}{|\vec{r} - \vec{r}\,'|} \psi_i(\vec{r}) +$$

$$\underbrace{- \frac{e^2}{4\pi\varepsilon_0} \sum_{i \neq j} \int d^3\vec{r}\,' \frac{\psi_j^*(\vec{r}\,')\,\psi_i(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^2} \psi_j(\vec{r})\,\delta_{m_{s_i}, m_{s_j}}}_{\text{ekarpen berria} \rightarrow \text{antisimetrizazioarekin lotuta}} = \varepsilon_i \psi_i$$

# KONFIGURAZIO ELEKTRONIKOA (EREMU ZENTRALAREN HURBILKETAN)

* Hurbilketa $\Rightarrow$ elektroi independenteak ditugu: $H = \sum_i H_i = \sum_i \left\{ \frac{-\hbar^2}{2m} \nabla_i + U_i(r_i) \right\}$

e⁻ bakoitzari dagokion hamiltondarra

e⁻ honek ikusten duen energia potentzial osoa zentrala dela suposatu (beste e⁻-en apantailatzea kontuan hartuz)

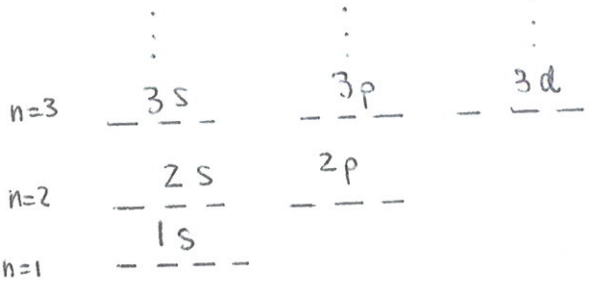
* Potentziala zentrala izanik $\{H, L^2, L_z\}$-ren aldiberekoak autofuntzioen oinarria aurkitu ahal izango dugu eta zenbaki kuantikoak $n$, $l$ eta $m_l$ izango dira $\Rightarrow$ $\Psi_{n,l,ml,ms}$ → spin-a sartzen badugu

* Batzuko oratoa $\Rightarrow$ $u_i(r_i)$ ezezaguna dugu $\Rightarrow$ hurbilketak

• Lehenengo hurbilketa: Demagun elektroien arteko elkarrekintzarik ez dagoela;

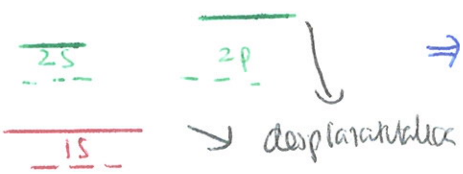
$$u_i(r_i) = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_i}$$ (Nukleoarekin elkarrekintza bakarrik) $\Rightarrow$ atomo hidrogenoideoaren emaitzak:

$$E_n = -\frac{Z^2 e^4 m^2}{2(4\pi\varepsilon_0)^2 \hbar^2} \cdot \frac{1}{n^2}$$ (endakatua)




Maila bakoitzean sartzen diren elektroien kopurua ezberdina da ($1s \to 2e^-$, $2p \to 6e^-$ ...)

Potentzial hau ez da benetakoa; apantailatze efektua hartu behar da kontuan $\Rightarrow$ energia mailak gora eramaten ("shift" → desplazatu) ; 2s eta 2p ez dira berdin desplazatuko, endakapena apurtuko du.




$\Rightarrow$ $l$ zenbat eta handiagoa izan, orduan eta gehiago desplazatuko da energia-maila

Atomo hidrogenoideoaren uhin-funtzioak: $\Psi_{n,l,ml,ms} \propto r^l$

$l\downarrow$ denean $\Rightarrow$ $r \to 0$ denean $r^l$ ekarpena handiagoa izango da eta nukleoarekin dagoen elektroiaren dagokion eharpena handiagoa izango da ; ≤ 0 denez energia maila baxuagoa izango da. ( l ↑ denean kontrakoa ⇒ gutxiago ikusiko dute nukleoaren eharpena eta apantailatze efektua nabarmenagoa izango da)

Hau hurbilketa da ⇒ adibidez 4s-rekin lotutako energia 3d-rekin lotutako energia baino baxuagoa da → ez daude ordenatuta atomo hidrogenoideen moduan (energia-mailen funtzioan elektroiak energia-maila ez bardinetan kokatzen joango dira) Nola eraiki konfigurazio elektronikoa?

## KONFIGURAZIO ELEKTRONIKOA: AUFBAU-REN PRINTZIPIOA

Konfigurazio elektronikoak gratzeko zenbait arau mnemoteknikoak daude ⇒ Aufbau-ren printzipioa (alemanez eraiki) ⇒ printzipio soilik fenomenologikoa

* Maila bat deskribatzeko n, l erabili → n berdina duten elektroiek geruza okupatzen dute eta l-ak azpigeruzekin lotuta daude
* Energia mailak nola okupatzen diren jakiteko nola ordenatzen diren jakin behar dugu ⇒ Aufbauren printzipioa erabili (Erwin Madelung-ek (batez ere) garatutakoa) ⇒

n+l zenbat eta txikiagoa izan, maila horren energia gero eta txikiagoa izango da (n+l arautzat ezagutzen da ere)

↳ Bi geruzen n+l balioa berdina bada n txikiena duenak izango du energia txikiagoa → arinago okupatuko da.

2 $e^-$ sartzen direlako

$1s^2$

$2s^2$ $2p^6$

$3s^2$ $3p^6$ $3d^{10}$

$4s^2$ $4p^6$ $4d^{10}$ $4f^{14}$

$5s^2$ $5p^6$ $5d^{10}$ . . .

* Adibidea ⇒ C → $Z = 6$ ⇒ konfigurazio elektronikoa lortzeko egin diagonalak. $1s^2\ 2s^2\ 2p^2$

Cu → $Z = 29$

$1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 4s^2\ 3d^9$

Arau hau ez da beti zuzena ⇒ Cu-ren kasuan, adibidez, 3d-ren energia 4s-rena baino baxuagoa da → ez da zuzena planteatu dugun konfigurazio elektronikoa: $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^{10}\ 4s^1$

Kasu honetan lehenengo hurbilketan pentsatuko genukeena laten da, n handitu ahala handitzen dela energia.

- Bestelako salbuespenak ⇒ trantsiziozko metalak (Cr, Pd, ...) → bestela nahiko aplikagarria da Aufbauren printzipioa.

AKOPLAMENDUAK ATOMOEN KONFIGURAZIO ELEKTRONIKOAK ZEHAZTEKO:

• Aplikatuko hurbilketak ⇒ elektroiak independenteak direla eta bakoitzari dagokion potentziala zentrala dela:

$$H = \sum_i \left\{ -\frac{\hbar^2}{2m} \nabla_i^2 + \underbrace{U_i(r_i)}_{} \right\}$$

$$\hookrightarrow U_i(r_i) = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_i} + \underset{\substack{\uparrow \\ \text{Apantailamendu efektiboa}}}{V_{ej}(r_i)}$$

beste $e^-$-en eragina

• Honenditik elektroi bakoitzari dagozkion energia-mailak lortzen dira, n, l, $m_l$ eta $m_s$ zenbaki kuantikoekin lotuta ⇒ honekin elementuen konfigurazio elektronikoa eraiki. Ad: $C\ (Z=6) \Rightarrow 1s^2\ 2s^2\ 2p^2$

Hala ere, azkenengo azpigeruza erabat beterik ez dagoenez ez dago erabat zehaztuta konfigurazio elektronikoa → energia mailen endekapenarekin lotuta ($L_z$ eta $S_z$-ren ekarpena ez da agertzen → $l$ eta $n$ berdina duten bi egoeren energia berdina da)

aukera bat

• C-ra itzuliz ⇒ 2p ⇒ $\underline{\uparrow\downarrow}$ $2p_1 \to m_l$ $\quad \underline{\quad}$ $2p_0 \to m_l$ $\quad \underline{\quad}$ $2p_{-1} \to m_l$

kontrako spinekin

Kokapen honen arabera atomo osoaren spina edo momentu angeluarra aldatu egingo da.

Hidrogenoarenaren ezberdina

Egoera hau ⇒ $\tilde{R}_{21}(r_1)\, Y_1^1(\theta_1, \varphi_1)\, R_{21}\, Y_1^1(\theta_2, \varphi_2) \otimes \frac{1}{\sqrt{2}}\left[|+,-\rangle - |-,+\rangle\right]$

$\overset{\text{osoak}}{S} = 0$ eta $L_z = 2\hbar$ ($L^2 = \hbar^2\, 2(2+1) = 6\hbar^2$ orduan)

Posibilitate gehiago! $\underline{\uparrow}$ $2p_1$ $\quad \underline{\uparrow}$ $2p_0$ $\quad \underline{\quad}$ $2p_{-1}$

Spin egoera berdina → egoera espaziala antisimetrikoa

$S=1$ eta $L=1$ (antisimetrikoa izateko → beste $l$-ren konprgonaziok simetrikoak dira)

Aukera gehiago daude, eta ikus daitekeenez, hauen funtzioan atomo osoaren $S$ eta $L$ aldatu egingo da (orien ospigsuna baino ez da izan behar kontuan hauek kalkulatzeko, azpimaskelan batzen 0 izaten delako)

- Hau aztertzeko hamiltondarrean eskerpen gehiago behar ditugu → elektroien arteko beste akoplamenduak aztertu behar ditugu ikusteko zein den atomoaren oinarrizko egoerari dagozkion $S$ eta $l$ osoak konfigurazio elektronikoa aipatutakoa izanda.

- Aurreko Hamiltondarrari $H_0$ deituko diogu ⇒ $\hat{H}_0 = \sum_i \left\{ -\frac{\hbar^2}{2m} \nabla_i^2 + U_i(r_i) \right\}$

Hamiltondar osoa ⇒ $\hat{H} = \hat{H}_0 + \underbrace{\left\{ \sum_{i<j} \frac{1}{4\pi\varepsilon_0} \frac{e^2}{|\vec{r}_i - \vec{r}_j|} - \sum_i V_{ef}(r_i) \right\}}_{\hat{H}_1 \text{ energia elektroniko erresiduala}} +$

hurbilduna

→ $H_0$-n dagoen eskerpena kentzeko

$\underbrace{\sum_i f_i(r_i) \hat{\vec{L}}_i \cdot \hat{\vec{S}}}_{H_2}$ → spin eta momentu angeluarraren arteko akoplamendua

Hau aztertzeko bi hurbilketa nagusi.

1) Atomo arinetan ($Z$ txikia, $Z \leq 30$) ⇒ $\hat{H}_1 >> \hat{H}_2$ → perturbatiboki aztertu $\hat{H}_2$ : LS akoplamendua (Russel-Saunders)

2) Atomo pisutsuetan (Z handia) ⇒ jj akoplamendua, $\hat{H}_2 >> \hat{H}_1$

perturbazioki aztertu $\hat{H}_1$.

## LS edo RUSSELL-SAUNDERS-en AKOPLAMENDUA: (Z ≤ 30)

Hamiltondar hurbildua ⇒ $H \approx \hat{H}_0 + \underbrace{\left\{ \sum_{i<j} \frac{1}{4\pi\varepsilon_0} \frac{e^2}{|\vec{r}_i - \vec{r}_j|} - \sum_i V_0(r_i) \right\}}_{\hat{H}_1} + \underbrace{\sum_i \xi_i(r_i)\, \hat{L}_i \cdot \hat{S}_i}_{\hat{H}_2}$

$\{\hat{H}_0, L_i^2, L_{zi}, \hat{S}_i^2, \hat{S}_{iz}\}$ 3-k BTMB osatzen dute

1/2 ($e^-$-ak)

$|n_i, l_i, m_{l_i}, s_i, m_{s_i}\rangle$

- Baina energia potentziala ez denez zentrala (beste elkarrekintzak kontuan hartuz)

$l_i$ eta $m_{l_i}$ ez dira zenbaki kuantiko onak.

$\vec{J} = (\vec{S} + \vec{L})$

$\{\hat{H}, \hat{L}^2, \hat{S}^2, \hat{J}^2, J_z\}$ 3-k BTMB osatzen dute → egoerak adierazteko erabiliko ditugun zenbaki kuantikoak: $|l\ s\ j\ m\rangle$ → energiak ordenatu beharko ditugu horien funtzioan.

fenomenologikoak

Horiei dagozkien energiak zehazteko arau batzuk daude: **Hunden arauak**

1- S zenbat eta handiagoa → energia gero eta txikiagoa: S↑ → E↓

S maximoari energia minimoa dagokio

* Uhin-funtzioaren antisimetrikotasunarekin lotuta

2- L zenbat eta handiagoa → energia gero eta txikiagoa: L↑ → E↓

Klasikoa


Beste elektroi bat badugu energia minimizatzeko bestearengandik ahalik eta urrunen egongo da → energia minimoa egonda atomoa, 2 $e^-$-a ahalik eta urrunen egongo da, zirkunferentzia baten ⇒ elektroi bien momentu angeluarrak batzerakoan momentu angeluarra maximoa egiten da.


(Azalpen kualitatiboa)

* Posibleak diren $l$ eta $s$-ren konbinazio guztiak ez dira posibleak ⇒ $\Psi = \Psi_{espaz.} \otimes \Psi_{spina}$

↳ Haien arabera J eraiki

$\Psi_{espaz.}$ ↳ L zehaztu

$\Psi_{spina}$ ↳ S zehaztu

Uhin-funtzioa antisimetrikoa izan behar denez L eta S ez dira edozein izan. Adibidez S maximoa izanda $\Psi_{spina}$ simetrikoa da eta

↗ $S = S_{max} - 1$ antisimetrikoa da

L-rekin gauza bera ⇒ $S_{max}$ eta $L_{max}$ ez da posiblea

3- L eta S konbinatuz ⟶ J-ren balio posibleak lortu.

* Azpigeruntzaren okupazioa erdia baino txikiagoa bada → J ↓ → E ↓
* Azpigeruntzaren okupazioa erdia baino handiagoa bada → J ↑ → E ↑

$\hat{H}_2$ perturbatiboki aztertu ⇒ $\langle l\ s\ j\ m | \hat{H}_2 | l\ s\ j\ m \rangle =$

$$g(l,s) \langle l\ s\ j\ m | \underbrace{\vec{L} \cdot \vec{S}}_{= \frac{1}{2}(J^2 - L^2 - S^2)} | l\ s\ j\ m \rangle = \frac{\hbar^2}{2} g(l,s) \cdot$$

$$[j(j+1) - l(l+1) - s(s+1)]$$

$j$ eta $j-1$ egoeren energien arteko aldea ($l$ eta $s$ berdinak izanda):

$$\Delta_{j-1,j} = \hbar^2 j\, g(l,s)$$

Adibideak: (Hund-en arauak aztertzeko)

## He (OINARRIZKO EGOERA)

He $(Z=2) \rightarrow 1s^2$ → ez dago endekapenik, azkenengo geruza erabat okupaturik dago ⇒ $S_z=0$ da ⇒ $S=0$ da

S egoeran egonik $L=0$ da (nahiz eta beste geruza baten egon, erabat okupatuta daudenez $L_z=0$ izango litzateke) ⇒ $J=0$

Hund-en arauak aplikatzeko beharrik ez dago.

## He (EGOERA KITZIKATUAK) $1s^1\ 2p^1$ edo $1s^1\ 2s^1$

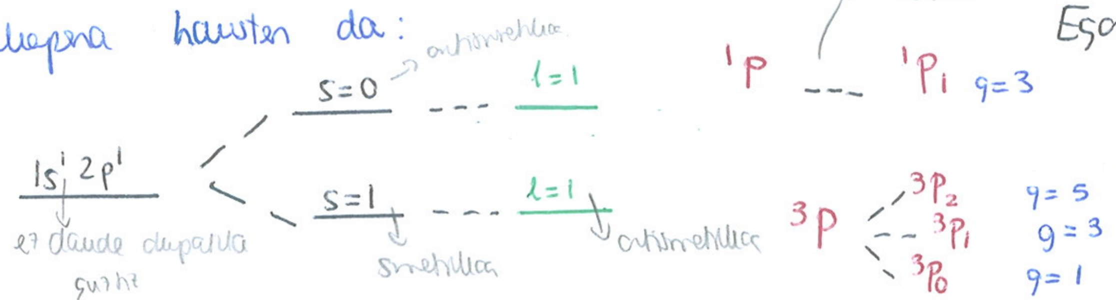
Hund-en arauak aplikatu: itzelezko endekapena.

Ad: $1s^1\ 2p^1$ ⇒ 2 $e^-$-en spina edozein izan daiteke eta p geruzan dagoen elektroiaren $m_l = -1, 0, 1$. ⇒ eremu zentraleko hurbilketan konfigurazio guztiek haien energia berdina, baita $e^-$ independenteen hurbilketan:

| | | |
|---|---|---|
| $2p_1$ ↑, $1s$ ↑ | $S_z = \hbar$, $L_z = \hbar$ ; | $2p_0$ ↓, $1s$ ↓ : $S_z = -\hbar$, $L_z = 0$ |

↳ $S = 1, 0$

Eremu zentralen hurbilketan sartzen ez diren elkarrekintzak kontuan hartuz endekapena hausten da:




Egoera espazialari dagokion elkarrekintza anti/simetrikoa

$1s^1 2s^1$ — $S=0$ — $l=0$ — $^1S_0$ $g=1$

$1s^1 2s^1$ — $S=1$ — $l=0$ — $^3S_1$ $g=3$

Hasieran 4 egoera baita

# C, KARBONOA: $C\ (Z=6) \rightarrow 1s^2\ 2s^2\ 2p^2$

2p azpigeruza ez dago guztiz okupatuta eta beraz endakapena da → hainbat konfigurazio egon daitezke (15 konfigurazio guztira)

Adibidez: $\uparrow\downarrow$ $2p_1$, — $2p_0$, — $2p_{-1}$ edo $\uparrow$ $2p_1$, $\uparrow$ $2p_0$, — $2p_{-1}$

Kasu hauei dagokien L eta S osoak ezberdinak dira

Endakapena hau hautsi egin behar da ⇒ Hunden arauak

Hauek aplikatzeko 2p azpimaila baino ez zaigu interesatzen, 1s eta 2s azpimailak erabat okupatuta daudenez haien S eta L osoa nuluak direlako.

$s_1 = s_2 = 1/2 \rightarrow S = s_1+s_2, \ldots |s_1 - s_2|$ ; $m_{l_1} = m_{l_2} = 1 \rightarrow l = 2, 1, 0$

$2p^2$ ($l_1 = l_2 = 1$):

- $S=0$: $l=0$ — $^1S$ — $^1S_0$ $g=1$; $\cancel{l=1}$; $l=2$ — $^1D$ — $^1D_2$ $g=5$
- $S=1$: $\cancel{l=0}$; $l=1$ — $^3P$ — $^3P_2$ $g=5$, $^3P_1$ $g=3$, $^3P_0$ $g=1$; $\cancel{l=2}$

guztira 15 konfigurazio

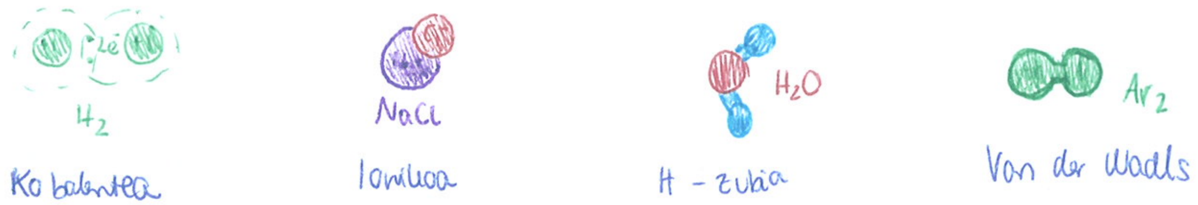
(simetrikoak direlako elkarren orbitala zein spin elkarrena) ↳ bi $e^-$-n egoera espaziala berdina izateko → ezinezkoa da

• ⇒ Dinamikoko egoera. → $e^-$-ak maila batetik bestera pasatu → hainbat salto egon daitezke hauen artean.

# MOLEKULAK

## SARRERA: MOLEKULAK

Atomoak neutroak dira orduan printzipioz hauen arteko elkarrekintzarik ez litzateke egongo. Hala ere, atomoek barne egitura dutenez honen berrantolaketa dela eta atomoen arteko lotura sortu daiteke eta horrela molekulak sortu. Lotura hauek ezberdinak dira eta ahulagoak edo sendoagoak izan daitezke. Adibidez, molekula sinpleetan zentratuz:




Batez ere lotura ioniko eta kobalenteetan zentratuko gara, kasu extremoak direlako:

Lotura kobalentea ⇒ antzeko atomoak lotu, ezaugarri kimiko antzekoak dituztenak

($HCl$, $CO_2$ ...

Lotura ionikoa ⇒ oso ezberdinak diren atomoak lotu

## AFINITATE ELEKTRONIKOA.

Afinitate elektronikoa atomo batek elektroi bat onartzeko duen joera da:

* Nola erakarri $e^-$-ak? Atomoek barne egitura bat dute eta hau berrantolatuz $e^-$ bat erakarri dezakete momentu dipolar bat sortuz ⇒ ioi negatiboa sortu.

✗ Nola kuantifikatu? X atomo bat badugu eta $e^-$ bat hurbiltzen badugu honako erreakzioa emango da: $X + e^- \rightarrow X^-$

Beraz afinitate elektronikoa honela izango da: $\underbrace{E(X^-) - E(X)}$ (eV)

↳ ioi negatiboaren energia atomoaren energiarekiko

energia askatzen dute $e^-$-a hartzean

Zenbat eta negatiboagoa izan orduan eta gero eta handiagoa izango du eta

balio absolutuan

orduan eta egonkorragoa izango da ioi negatiboa. Afinitate handiena dutenak halogenoak dira (−3 eV inguruan), egitura elektronikoko azkeneko azpigeruza $np^5$ delako $\Rightarrow$ 1 $e^-$ hartuz azkeneko azpigeruza beteko litzateke.

Honela, azkeneko azpigeruza beteta dagoenean (gas nobleak, lur alkalinoak...) ez dago $e^-$ baten beharrik eta beraz afinitate elektronikoa nulua da.

Afinitate elektronikoa taulan eskuinerantz handitzen da, halogenoetarantz

↓ balio absolutua!

zenbat eta hurbilago.

## IONIZAZIO-ENERGIA:

Ionizazio-energia atomo bati eman behar zaion energia minimoa da $e^-$ bat askatzeko (≈ afinitate elektronikoaren "kontrakoa").

$$X \rightarrow X^+ + 1e^-$$

↓ eman behar zaion energia

*Ionizazio - energia txikiena dutenak Li, Na, K, Rb, Cs ... dira ⇒ alkalinoak: azkenengo azpigeruzan $ns^1$ dutelako ⇒ ez dago konfiguratzeko lotura handirik atomoaren nukleo eta $e^-$ haren artean eta beraz oso energia baxua behar da askatzeko.

* Ionizazio - energia handiena dutenak He, Ne, Ar, Kr, Xe eta Rn dira ⇒ gas nobleak: azkenengo azpigeruzan $np^6$ dutelako ⇒ azpigeruza guztiz beteta dute, ez dute $e^-$ bat askatzeko beharrik.

Taulan eskuinerantz handitzen da ionizazio energia eta gorantz.

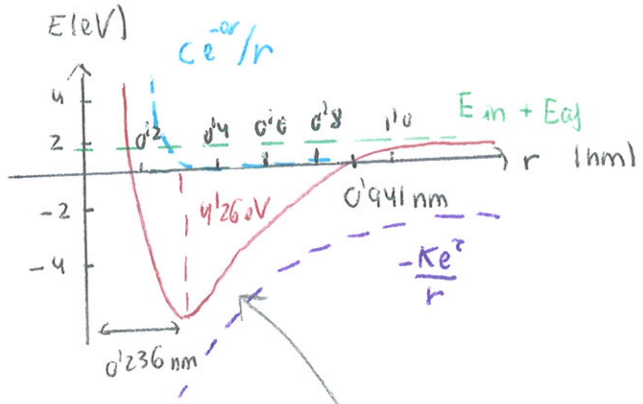
↑ →

## LOTURA-IONIKOA:

Lotura-ionikoa lortzeko ionizazio - energia txikiko eta afinitate - elektroniko handiko (balio absolutuan) bi atomo behar ditugu (Adibidez NaCl)

$$\begin{cases} Cl \Rightarrow \text{halogeno bat, afinitate elektroniko handikoa} ; \; E_{af}(Cl) = -3'62 \, eV \\ Na \Rightarrow \text{alkalino bat, ionizazio - energia baxukoa} ; \; E_{ion}(Na) = 5'14 \, eV \end{cases}$$

Na-ri $e^-$ bat kenduko diogu $Na^+$ sortuz eta $e^-$ hori Cl-ri eman $Cl^-$ sortuz (horrela energia askatu, $E_{af}$) ⇒ bi ioi hauek oso aldenduta egonda prozesu honetan atomoek duten energia $E_{ion} + E_{af}$ izango da (bati energia eman behar izan zaio eta besteak askatu egin du) ⇒ $E = 1'52$ eV (bi ioi horien energia)




Hala ere, bi ioiek karga kontrakoa dutenez erakarri egingo dira:

- $E = E_{ion} + E_{af} - \frac{Ke^2}{r} + C\frac{e^{-ar}}{r}$

Coulomb-en energia

Pauliren aldaraparen energia

Gero eta gehiago hurbilago egotean orduan eta baxuagoa izango da energia ⇒ egonkorragoa izango da bi ioi izatea bi atomo neutroak baino. Hala ere, ezin dugu nahi dugun beste hurbildu ioi bakoitzak barne egitura duenez asko hurbiltzean $e^-$-n artekoak aldaraparak egingo direlako: $Na + Cl^-$ ⇒ Pauliren aldarapena

Sistemak energia minimoranz joko du eta beraz ioien arteko distantzia energia minimizatzen duen distantzia izango da: $d_{min} = 2'36\,Å \equiv r_0$ eta $E_{min} = -4'26$ eV ⇒ Molekulari eman behar zaion energia bi atomoak banatzeko (disoziazio energia)

## MOLEKULA BATEN HAMILTONDARRA eta BORN-OPPENHEIMER-EN HURBILKETA:

nukleoak geldi

Born-Oppenheimer-en hurbilketa; elektroi eta nukleoen higidurak independenteki aztertu;

$$\hat{H} = \sum_n \frac{\hat{p}_n^2}{2m_e} + \sum_N \frac{\hat{P}_N^2}{2m_N} + V(\{\vec{r}_n\}, \{\vec{R}_N\})$$

$e^-$-en energia zinetikoa

nukleoen energia zinetikoa

energia potentzial osoa

Autofuntzioak eta autobalioak lortzea izango da gure helburua eta horretarako hurbilketak aplikatu beharko ditugu

- $e^-$ eta nukleoen gaineko indarra antzekoa da (ingurune beran daude) → momentu linealak magnitude ordena berekoak dira baina diferentzia masan datza ( $m_N \simeq 2000\, m_e$ ) $\Rightarrow$ $T_e \gg T_N$ izango da eta arbuiatu egingo dugu $\Rightarrow$ $\hat{H} = \sum_n \frac{\hat{\vec{P}}_n^2}{2m_e} + V(\{\vec{r}_n\}, \{\vec{R}_N\})$

$e^-$-n autofuntzioen ekuazioa $\Rightarrow$ $\left[-\sum_n \frac{\hbar^2}{2m_e}\nabla_n^2 + V(\{\vec{r}_n\},\{\vec{R}_N\})\right]\Psi_K(\{\vec{r}_n\},\{\vec{R}_N\}) =$

$E_K\,\Psi_K(\{\vec{r}_n\},\{\vec{R}_N\})$

↳ $\vec{R}_N$ parametroen menpekoa ere

$V$ $\vec{R}_N$-ren menpekoa denez $\Psi$ haren menpekoa izango da ere (guk parametrotzat hartuko ditugu, balio finko batzuk hartu)

Baina hau ebatzi ahal izateko zein parametro hartu beharko ditugu?

- Lehenengo helburua $e^-$-en oinarrizko egoera kalkulatzea izango da eta oinarrizko energia: $E_0(\{\vec{R}_N\})$

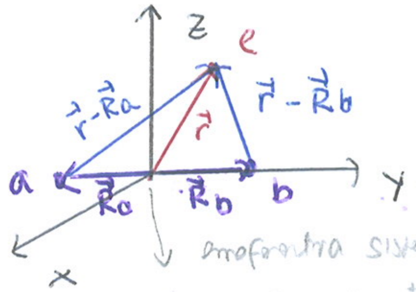
Hau $\{\vec{R}_N\}$ parametroekiko kalkulatu egingo dugu eta minimizatu horrela $\Rightarrow$ minimo hori nukleoen arteko oreka-distantzia izango da $\Rightarrow$

oinarrizko egoeraren uhin-funtzioa eta energia kalkulatu ahal izango ditugu

↳ Nukleoen posizioen espazioan $E_0$ kalkulatu

Nola inplementatu metodo hau?

# LOTURA KOBALENTEA: $H_2^+$ molekula




Erreferentzia sistemaren jatorria bi $p^+$-en arteko erdiko puntuan

Molekularik sinpleena ⇒ $H_2^+$ $(2p^+, 1e^-)$

Hamiltondarra Born-Oppenheimer-en hurbilketan:

$$\hat{H}\Psi(\vec{r};\vec{R}_a,\vec{R}_b) = \left\{-\frac{\hbar^2}{2m}\nabla_r^2 - \frac{e^2}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{R}_a|}+\frac{1}{|\vec{r}-\vec{R}_b|}\right]\right\}\Psi(\vec{r};\vec{R}_a,\vec{R}_b) =$$

$$E(\vec{R}_a,\vec{R}_b)\Psi(\vec{r};\vec{R}_a,\vec{R}_b)$$

E. sistemaren jatorria 2 $p^+$-en arteko erdiko puntuan kokatu dugunez hamiltondarrak jatorriarekiko inbertsio simetria izango du. Halaber, 2 $p^+$-ak Y ardatzean kokatuko ditugu.

Halaber, $\hat{H}$-aren 2 $p^+$-en arteko energia potentzialaren ekarpena izango dugu baina $\vec{R}$-ak balio finkoak direnez konstante bat izango da ⇒ E-n egingo dute ekarpena, ez du uhin-funtzioa aldatuko. Beraz molekula osoaren energia kalkulatu nahi badugu balio hauek gehitu beharko dizkiogu E-ri (definizioz $e^-$-aren energia soilik)

Goiko adierazpenak badu emaitza analitiko eta zehatza baina guk hurbilketak egingo ditugu.

a eta b protoiak: oso aldenduta ⟶ $e^-$-a bat edo bestearen inguruan egongo da; atomikoko egoeran badago 1s-ko egoeran


$\Psi_a^{1s}$


$\Psi_b^{1s}$

Orduan, uhin-funtzioa hurbilduz bi hauen konbinazio bat dela esan dezakegu (oinarrizko egoera):

$$\psi(\vec{r};\vec{R}_a,\vec{R}_b)=\alpha\,\psi_a^{1s}(\vec{r},\vec{R}_a)+\beta\,\psi_b^{1s}(\vec{r},\vec{R}_b)$$

( Gogoratu $\Rightarrow$ $\psi_{a,b}^{1s} = e^{-|\vec{r}-\vec{R}_{a,b}|/a_0} \Big/ (\pi a_0^3)^{1/2}$ )

Halaber, hamiltondarrak inbertsio simetria duenez autofuntzioak simetrikoak edo antisimetrikoak izango dira $\Rightarrow$ $\alpha = \pm\beta$

Bi aukera:
$$\begin{cases} \psi(\vec{r};\vec{R}_a,\vec{R}_b)= \alpha\left[\psi_a^{1s}(\vec{r};\vec{R}_a)+\psi_b^{1s}(\vec{r};\vec{R}_b)\right] & \text{simetrikoa} \\ \psi(\vec{r};\vec{R}_a,\vec{R}_b)= \alpha\left[\psi_a^{1s}(\vec{r};\vec{R}_a)-\psi_b^{1s}(\vec{r};\vec{R}_b)\right] & \text{antisimetrikoa} \end{cases}$$

Beraz $\Rightarrow$ $\psi^{\pm}(\vec{r};\vec{R}_a,\vec{R}_b)=\alpha(\psi_a\pm\psi_b)$ eta $\alpha\in\mathbb{R}$

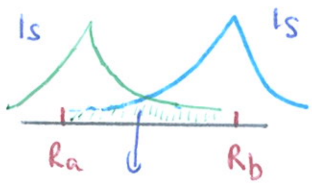
hartuz eta normalizatuz:

$$\langle\psi^{\pm}|\psi^{\pm}\rangle=\alpha^2\langle\psi_a\pm\psi_b|\psi_a\pm\psi_b\rangle=\alpha^2\Big[\underbrace{\langle\psi_a|\psi_a\rangle}_{=1}\pm$$

$$\underbrace{\langle\psi_b|\psi_a\rangle\pm\langle\psi_a|\psi_b\rangle}_{\text{berdinak};\ S}+\underbrace{\langle\psi_b|\psi_b\rangle}_{=1}\Big]=\alpha^2(2\pm2S)=1$$

$$\boxed{\alpha=\frac{1}{\sqrt{2(1\pm S)}}}$$

Uhin-funtzio bera puntu ezberdinetan zentratuta:




bien arteko solapamendua

$\langle\psi_b|\psi_a\rangle = S$ (solapamendu integrala)

Zein da egoera horren energia? (Oinarrizko energia)

$$\bullet\ E^{\pm} = \langle \psi^{\pm} | \hat{H} | \psi^{\pm} \rangle = \frac{1}{2(1\pm S)} \{ \langle \psi_a \pm \psi_b | \hat{H} | \psi_a \pm \psi_b \rangle \} =$$

$$\frac{1}{2(1\pm S)} \{ \langle \psi_a | \hat{H} | \psi_a \rangle \pm \langle \psi_b | \hat{H} | \psi_a \rangle \pm \langle \psi_a | \hat{H} | \psi_b \rangle + \langle \psi_b | \hat{H} | \psi_b \rangle \}$$

berdinak

$H_{ab} = \langle \psi_b | \hat{H} | \psi_a \rangle$

berdinak dira → $H_{aa} = \langle \psi_a | \hat{H} | \psi_a \rangle$

$$\boxed{E^{\pm} = \frac{H_{aa} \pm H_{ab}}{1 \pm S}}$$

Baina zein da oinarrizkoena? Hurrengo ataletan

# LINEAR COMBINATION OF ATOMIC ORBITALS (LCAO)

$H_2^+$ molekula aztertzeko erabili dugun metodoa orokorragoa den metodo baten kasu partikularra da ⇒ LCAO metodoa (John Lennard-Jones, 1929)

Atomo baten orbital atomikoek oinarri bat osatzen dute ⇒ edozein funtzio hauen konbinazio lineal moduan adierazi daiteke ⇒ molekula baten uhin-funtzio elektronikoa (elektroiarena) a eta b-n (2 protoiak) zentratutako orbital atomiko diferenteen konbinazio lineal moduan adierazi daiteke.

$$\psi = c_1^a \psi_a^{1s} + c_2^a \psi_a^{2s} + \ldots + c_1^b \psi_b^{1s} + c_2^b \psi_b^{2s} + \ldots$$

↓ beste guztiak ↓ beste guztiak

Dimentsioa infinitua denez garapen hori moztu egiten da, ekarpen batzuk arbiagonalak izango dira $\Rightarrow$ atomoaren arabera lotu baten edo beste mota $\Rightarrow$ H-ren kasuan 1s-rekin geratu bagarenik (zehatzagoa izatea nahi badugu termino gehiago sortuko genituzke)

* 1. hurbilketa $\Rightarrow$ $\{\psi_a^{1s}, \psi_b^{1s}\}$ oinarri finitua

Oinarri honetan garatutako Hamiltondarraren adierazpen matriziala:

- Arazoa $\Rightarrow$ oinarria ez da ortogonala $\langle \psi_a^{1s} | \psi_b^{1s} \rangle \neq 0$

Ortogonala izango balitz, lortuko genituzkeen autofuntzioak

$$\psi = c_1 \psi_a^{1s} + c_2 \psi_b^{1s} \text{ izango lirateke eta } H = \begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix}$$

non $H_{aa} = \langle \psi_a^{1s} | \hat{H} | \psi_a^{1s} \rangle = H_{bb}$ eta $H_{ab} = H_{ba}$

Ebatzi beharreko ekuazioa: $\begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \varepsilon \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

- Ortogonala ez denez $\Rightarrow$ prozedura ezberdina:

$$\begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \varepsilon \underbrace{\begin{pmatrix} 1 & \underset{S}{\overset{\langle \psi_a | \psi_b \rangle}{\parallel}} \\ \underset{S}{\overset{\langle \psi_b | \psi_a \rangle}{\parallel}} & 1 \end{pmatrix}}_{\tilde{S}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow$$

(solapamendu integrala)

$$\begin{pmatrix} H_{aa} - \varepsilon & H_{ab} - \varepsilon S \\ H_{ab} - \varepsilon S & H_{aa} - \varepsilon \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \varepsilon^{\pm} = \frac{H_{aa} \pm H_{ab}}{1 \pm S}$$

energia kopurua orbital kopuruaren berdina

$$\psi^{\pm} = c\left(\psi_a^{1s} \pm \psi_b^{1s}\right) = \frac{1}{\sqrt{2(1\pm S)}}\left(\psi_a^{1s} \pm \psi_b^{1s}\right)$$

Aurreko metodoaren emaitza berdina

Orain orbital gehiago sartu ahal izango genituzke ⇒ matrizea handiagoa ⇒ emaitza gehiago eta zehatzagoak

## $H_2^+$ MOLEKULAREN OINARRIZKO ENERGIA:

Aurreko metodotik: $\psi^{\pm}(\vec{r};\vec{R}_a,\vec{R}_b) = \frac{1}{\sqrt{2(1\pm S)}}\left[\psi_a^{1s}(\vec{r};\vec{R}_a) \pm \psi_b^{1s}(\vec{r};\vec{R}_b)\right]$

$$E^{\pm} = \frac{H_{aa} \pm H_{ab}}{1 \pm S}, \quad S = \langle \psi_a^{1s} | \psi_b^{1s} \rangle \quad \text{eta}$$

$$H_{aa} = \langle \psi_a^{1s} | \hat{H} | \psi_a^{1s} \rangle, \quad H_{ab} = \langle \psi_a^{1s} | \hat{H} | \psi_b^{1s} \rangle$$

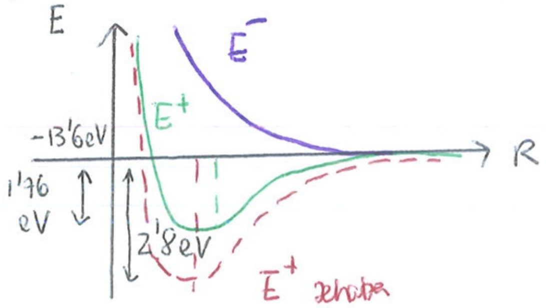
Integralak eginez:

* $S = \left(1 + \frac{R}{a_0} + \frac{R^2}{3a_0^2}\right) e^{-R/a_0}$ $\quad R \rightarrow$ Bi protoien arteko distantzia $|\vec{R}_a - \vec{R}_b|$

* $H_{aa} = E_0^H + \frac{e^2}{4\pi\varepsilon_0 R}\left(1 + \frac{R}{a_0}\right) e^{-2R/a_0}$ $\quad E_0^H \rightarrow$ H atomoaren oinarrizko energia

* $H_{ab} = \frac{e^2}{4\pi\varepsilon_0 a_0}\left(1 + \frac{R}{a_0}\right) e^{-R/a_0} + S\left(E_0^H + \frac{e^2}{4\pi\varepsilon_0 R}\right)$

Orduan ⇒




$E^+$-ek ⇒ minimoa
$E^-$-ek ⇒ minimorik ez
$R \rightarrow \infty$ ⇒ balio berdinera hurbildu

- $R \to \infty$ denen balio berdina ematen dute distantzia oso handietan oso aldenduta daudenez batean edo bestean dagoelako elektroia ⇒

$$E = -13'6 \text{ eV}$$ (batean zentratuta dagoenean)

↘ elkarrekintzarik ez

- $R$ txikiagotzean $E^+$-ek minimo bat du; minimoa dagoen $R$ izango da protoien arteko distantzia ⇒ gure hurbilketan $1'3$ Å

- Protoiak banatzeko eman behar dugun energia (disoziazio energia) minimoaren sakonera da → $1'76$ eV ($E_0$-tik minimora dagoen tartea)

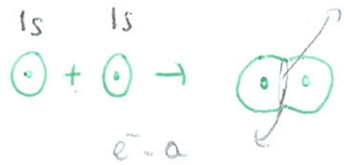
  ↗ $R \to \infty$ deneko energiarekiko

- Beraz ⇒ oinarrizko egoera egoera simetrikoa da eta $R = 1'3$ Å izango da. Gainera disoziazio energia $1'76$ eV da.

↗ H honek emaitza analitikoa du

Benetan emaitza zehatza ezberdina da. ⇒ $R = 1'06$ Å eta disoziazio energia $2'8$ eV

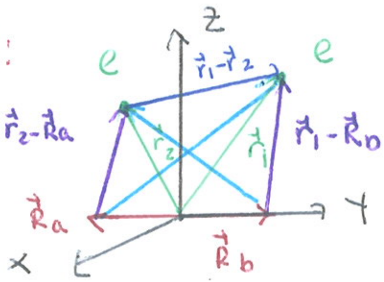
Hala ere emaitzak hurbilketarako ez dira oso txarrak.

Simetrikoa ⇒ $\psi^+$  ; 


Hemen uhin-funtzioa ez da nulua

lotura handiagoa ⇐ $e^-$-a tartekatu egiten da 2 protoien artean

## $H_2$ MOLEKULA:




$$\hat{H} = -\frac{\hbar^2}{2m}\left(\nabla_{\hat{r}_1}^2 + \nabla_{\hat{r}_2}^2\right) + \frac{e^2}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{R}_a - \vec{R}_b|} + \frac{1}{|\vec{r}_1 - \vec{r}_2|} + \right.$$

$$\left. - \frac{1}{|\vec{r}_1 - \vec{R}_a|} - \frac{1}{|\vec{r}_1 - \vec{R}_b|} - \frac{1}{|\vec{r}_2 - \vec{R}_a|} - \frac{1}{|\vec{r}_2 - \vec{R}_b|}\right]$$

$H_2^+$ molekularen hamiltondarorekin lotu $\Rightarrow$ $\hat{H} = \underbrace{\hat{H}_{H_2^+(1)} + \hat{H}_{H_2^+(2)}}_{\hat{H}_0} +$

1.e⁻-a (2 p⁺-en arteko V sortuta)

$$\underbrace{\frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}_1-\vec{r}_2|} - \frac{1}{|\vec{R}a-\vec{R}b|}\right)}_{\text{elkarrekintza hau oso txikia}}$$

2 aldiz sortu dugulako

$\Rightarrow$ perturbatiboki artertu ($\hat{H}$aur)

- $H_0$-ren emaitzak $\hat{H}_{H_2^+(1)} + \hat{H}_{H_2^+(2)}$-renak

$E_-$ — antisimetrikoak

$E_+$ $\uparrow\downarrow$ simetrikoak

↳ oinarrizko egoeran 2 e⁻-ak hemen (spin erdierdiaren)

Oinarrizko egoera $\Rightarrow$

simetrikoa

$$|\Psi_0^{H_2}\rangle = |\psi^+(\vec{r}_1;\vec{R}a,\vec{R}b)\,\psi^+(\vec{r}_2;\vec{R}a,\vec{R}b)\rangle \otimes \overset{s\ \ m_s}{|0,0\rangle}$$

antisimetrikoa $\frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$

$$E_0^0 = 2E^+$$

- Perturbazioa sekitu $\Rightarrow$ $E_{0\,(R)} = 2E^+ + \langle \Psi_0^{H_2} | \hat{H}_{elk} | \Psi_0^{H_2} \rangle$

Hau minimizatu eta $R$ orekako distantzia lortu $\Rightarrow$ $R_0 = 0'85\ \mathring{A}$

$E_{disoziazioa} = 2'7\ eV$

* Emaitza esperimentalak $\Rightarrow$ $R_0 = 0'74\ \mathring{A}$ eta $E_{disoziazioa} = 4'75\,eV$

## $H_2$ MOLEKULA AZTERTZEKO EGINDAKO HURBILKETAREN ARAZOA:

$$|\Psi_0^{H_2}\rangle = |\underbrace{\psi^+(\vec{r}_1;\vec{R}a,\vec{R}b)}_{\frac{1}{\sqrt{2(1+S)}}\left(\psi_a^{1s}(\vec{r}_1;\vec{R}a) + \psi_b^{1s}(\vec{r}_2;\vec{R}b)\right)}\psi^+(\vec{r}_2;\vec{R}a,\vec{R}b)\rangle \otimes \overset{s\ \ m_s}{|0,0\rangle} \Rightarrow \text{singletea}$$

Oinarrizko egoera deskribatzeko uhin-funtzioa $\Rightarrow$ hurbilketa gogor bat egin dugu.

Egoera espaziala dagokion uhin-funtzioa garatuko dugu:

$$\psi_0^{H_2} \propto \Big[ \underbrace{\psi_a^{1s}(\vec{r}_1)\,\psi_b^{1s}(\vec{r}_2) + \psi_b^{1s}(\vec{r}_1)\,\psi_a^{1s}(\vec{r}_2)}_{\text{lotura kobalentea}} \Big] + \Big[ \underbrace{\psi_a^{1s}(\vec{r}_1)\,\psi_a^{1s}(\vec{r}_2)}_{\text{biak a-n egoteko probabilitatea}} + \underbrace{\psi_b^{1s}(\vec{r}_1)\,\psi_b^{1s}(\vec{r}_2)}_{\text{lotura ionikoa}} \Big]$$

1. $e^-$-a a-n zentratuta eta 2. $e^-$-a b-n zentratuta

1 $e^-$-a b-n zentratuta eta 2 $e^-$-a a-n zentratuta

egoteko probabilitatearekin lotuta.

($e^-$ biak protoi batean zentratuta → partekatzen ez dira)

$R \to \infty \implies P(H^+ \; H^-)$ edo $P(H \; H)$ berdinak dira

batek 2 $e^-$ — besteak 0 — biek protoi bat

- Horela ez daukagu zentzu handirik $\Rightarrow P(H\;H)$ handiagoa izan beharko

litzateke, $e^-$-ak bien artean banatuta badaude

Lotura ionikoarekin lotutako eragpenaren pisua handiegia da!

- Beste hurbilketa bat (Balentzia lotura hurbilketa):

$$\psi_0^{BL} \propto \left[\psi_a^{1s}(\vec{r}_1)\,\psi_b^{1s}(\vec{r}_2) + \psi_b^{1s}(\vec{r}_1)\,\psi_a^{1s}(\vec{r}_2)\right] + \lambda\left[\psi_a^{1s}(\vec{r}_1)\,\psi_a^{1s}(\vec{r}_2) + \psi_b^{1s}(\vec{r}_1)\,\psi_b^{1s}(\vec{r}_2)\right]$$

$\lambda$? $\Rightarrow$ Metodo bariazionala aplikatu $\Rightarrow E_0$ minimizatu $\lambda$-rekiko:

$$\lambda = \frac{1}{6}$$

(Gure hurbilketan baino askoz txikiagoa)

## BORN-OPPENHEIMER-en HURBILKETA (jarraipena)

$$\left[\sum_n \frac{-\hbar^2}{2m}\nabla_n^2 + V(\{\vec{r}_n\};\{\vec{R}_N\})\right]\psi_K(\{\vec{r}_n\};\{\vec{R}_N\}) = E_K(\{\vec{R}_N\})\,\psi_K(\{\vec{r}_n\};\{\vec{R}_N\})$$

Nukleoekin lotutako energia zinetikoa arbuiatu egin genuen $\Rightarrow$ oran nukleoen higidura aztertuko dugu.

• Born-Oppenheimer-en hurbilketan $\{\vec{R}_N\}$-ak parametrotzat hartzen genituen.

Elektroien posizio bakoitzeak kontuan hartzen baditugu uhin-funtzio elektronikoek

oinarri bat osatzen dute: $\{\psi_K\}$ (Hau ortonormala aukeratu)

Molekula osoei (nukleoa barne) dagozkien uhin-funtzioak oinarri honetan garatu ahal

izango ditugu; Uhin-funtzio molekularrak:

osoa

funtzioak

bi aldagaiak

$$\Psi(\{\vec{r}_n\},\{\vec{R}_N\}) = \sum_K \underbrace{\phi_K(\{\vec{R}_N\})}_{=\,(\psi_K,\,\Psi)=\int d^3\{\vec{r}_n\}\,\psi_K\,\Psi} \psi_K(\{\vec{r}_n\};\{\vec{R}_N\})$$

Nukleoaren higiduraren ekarpenak

$\{\vec{R}_N\}$-ren integralik ez dugu egiten $\Rightarrow$ $\phi_K$ haren menpekoa

Molekula osoaren autofuntzioei dagokien ekuazioa:

$$\left[-\sum_n \frac{\hbar^2}{2m}\nabla_n^2 - \sum_N \frac{\hbar^2}{2m_N}\nabla_N^2 + V(\{\vec{r}_n\},\{\vec{R}_N\})\right]\Psi = E\,\Psi$$

→ osmaketa gorpena haren ordezkatu

* Kontuan hartu $\psi_K$-k betetzen duen ekuazioa!

parametro bezala

$$\sum_K \left[-\sum_N \frac{\hbar^2}{2m_N}\nabla_N^2 + E_K(\{\vec{R}_N\})\right]\phi_K\,\psi_K = E\sum_K \phi_K\,\psi_K$$

Nolabait e⁻-en energiak leun kontsideratzen ditugu

• Born-Oppenheimerren hurbilketa nagusia: $\nabla_N^2(\phi_K\psi_K) \simeq \psi_K\nabla_N^2\phi_K$ — $\psi_K$-ren gainetik denbarbide arbuiatu

$\nabla_N^2\psi_K$ $\nabla_N^2\phi_K$-rekiko arbuiatu dugu ($\nabla^2$ kurbadurarekin lotua $\Rightarrow$

$\psi_K$-k duen kurbadura $\phi_K$-k duena baino txikiagoa da)

↓ $\Lambda^{\phi_K}$ $\phi_K$ pikutsuagoa da

Beraz $\Rightarrow \displaystyle\sum_K \Big[ -\sum_N \frac{\hbar^2}{2m_N} \psi_K \nabla_N^2 \phi_K + E_K(\{\vec{R}_N\})\phi_K \psi_K \Big] = E \sum_K \phi_K \psi_K$

Hau beste modu batean adierazteko ezkerreko eta eskuineko aldeak ezkerretik $\psi_j$ batekin biderkatuko dugu eskalarki.

ortonormalak

$$\Big(\psi_j, \sum_K \Big[ -\sum_N \frac{\hbar^2}{2m_N} \psi_K \nabla_N^2 \phi_K + E_K(\{\vec{R}_N\})\phi_K \psi_K \Big]\Big) = \Big(\psi_j, E\sum_K \phi_K \psi_K\Big) =$$

$$-\sum_N \frac{\hbar^2}{2m_N} \nabla_N^2 \phi_j + E_j(\{\vec{R}_N\})\phi_j = E\phi_j \Rightarrow \phi_j \text{ funtzioak betetzen duten ekuazioa}$$

nukleoen higidura

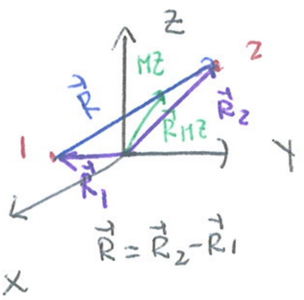
nukleoekin lotutako energia potentzial efektibo osoa (e⁻-en ekarpena sartuta)

Higidura nukleoen hamiltondarraren ekuazioa. Gainera $j$ hori egoera elektronikoekin lotuta dago $\Rightarrow$ interesatuko zaiguna: zein da higidura nukleoena elektroiak oinarrizko egoeran daudenean? $j=0$-n.

Ebatzi beharreko ekuazioa

$$j=0 \Rightarrow -\sum_N \frac{\hbar^2}{2m_N} \nabla_N^2 \phi_0 + E_0(\{\vec{R}_N\})\phi_0 = E\phi_0$$

# NUKLEOEN HIGIDURA MOLEKULA DIATOMIKO BATEAN




$\vec{R} = \vec{R}_2 - \vec{R}_1$

$$E_0(\vec{R}_1, \vec{R}_2) = E_0(|\vec{R}_2 - \vec{R}_1|) = E_0(R)$$

$$\Big[ -\frac{\hbar^2}{2m_1}\nabla_{\vec{R}_1}^2 - \frac{\hbar^2}{2m_2}\nabla_{\vec{R}_2}^2 + E_0(R)\Big]\phi_0(\vec{R}_1, \vec{R}_2) = E\phi_0(\vec{R}_1, \vec{R}_2)$$

bi nukleoen arteko potentzial efektiboa (e⁻-en ekarpena ere sartuta)

$\phi_0(\vec{R}_1, \vec{R}_2) \Rightarrow$ e⁻-en oinarrizko egoeran nukleoen higidura

Aldagai aldaketa: (2 partikula direnean beti berdina)

$\vec{R} = \vec{R}_2 - \vec{R}_1$ eta $\vec{R}_{MZ} = \dfrac{m_1\vec{R}_1 + m_2\vec{R}_2}{m_1+m_2}$ posizio erlatiboaren moduluaren menpekoa soilik

Orduan $\Rightarrow$ $\left[-\dfrac{\hbar^2}{2M}\nabla^2_{\vec{R}_{MZ}} - \dfrac{\hbar^2}{2\mu}\nabla^2_{\vec{R}} + E_0(R)\right]\phi_0(\vec{R}_{MZ},\vec{R}) = E\,\phi_0(\vec{R}_{MZ},\vec{R})$

masa osoa

masa laburbildua: $\mu = \dfrac{m_1 m_2}{m_1+m_2}$

$*$ Uhin-funtzioa banangarria (Hamiltondarra banangarria da):

$$\phi_0(\vec{R}_{MZ},\vec{R}) = \phi_0^{MZ}(\vec{R}_{MZ})\,\tilde{\phi}_0(\vec{R})$$

a) $-\dfrac{\hbar^2}{2M}\nabla^2_{\vec{R}_{MZ}}\phi_0^{MZ}(\vec{R}_{MZ}) = E^{MZ}\phi_0^{MZ}(\vec{R}_{MZ}) \longrightarrow E^{MZ} = \dfrac{\hbar^2 K^2}{2M}$, $\phi_0^{MZ}(\vec{R}_{MZ}) \propto e^{i\vec{K}\cdot\vec{R}_{MZ}}$

↳ translaziozko lekualdaketa

Partikula askea $\Rightarrow$ uhin laua $(E = E^{MZ} + \tilde{E})$

b) $-\dfrac{\hbar^2}{2\mu}\nabla^2_{\vec{R}}\tilde{\phi}_0(\vec{R}) + E_0(R)\tilde{\phi}_0(\vec{R}) = \tilde{E}\,\tilde{\phi}_0(\vec{R})$ $\rightarrow$ barne-higiduraren lekualdaketa

Beti aukeratu dezakegunez erreferentzia sistemaren jatorria MZ-on jartzea

$E^{MZ} = 0$ izango litzateke.

energia potentzial zentrala

Ekuazioa $\Rightarrow$ $-\dfrac{\hbar^2}{2\mu}\nabla^2_{\vec{R}}\tilde{\phi}_0(\vec{R}) + E_0(R)\tilde{\phi}_0(\vec{R}) = \tilde{E}\,\tilde{\phi}_0(\vec{R})$

errotazio momentu angeluarra

Laplacearra koordenatu esferikoetan $\Rightarrow$ $-\dfrac{\hbar^2}{2\mu}\cdot\dfrac{1}{R}\partial_R^2 R + \dfrac{\hat{J}^2}{2\mu R} = -\dfrac{\hbar^2}{2\mu}\nabla^2_{\vec{R}}$

$\{\tilde{H}, \hat{J}^2, \hat{J}_z\}$-ren aldiberako autofuntzioak hartu; $|\nu, J, m_J\rangle$

($\phi_0$ zenbaki kuantikoen menpekoa $\Rightarrow$ $\phi_0 \equiv \phi_{0,\nu,J,m_J}$)

↳ zenbaki kuantikoa ere, $e^-$-en egoera adierazten duelako

* Kasu honetan, energiak ez dute $m_J$-ren menpekotasunik izango, Hamiltondarean $J_z$-ren eragilerik ez dagoelako $\Rightarrow$ egoerek $m_J$-ren endakapena izango dute.

$\{J_z, \vec{J}^2\}$-ren autofuntzioak

$$\tilde{\phi}_{0,\nu,J,m_J} = \frac{1}{R}\tilde{R}_{0,\nu,J}(R)\, Y_J^{m_J}(\theta,\varphi)$$ saiatu

- Hau kontuan hartuz:

$$\left[-\frac{\hbar^2}{2\mu}\frac{1}{R}\partial_R^2 R + \frac{\hat{\vec{J}}^2}{2\mu R^2} + E_0(R)\right]\tilde{\phi}_{0,\nu,J,m_J}(\vec{R}) = \tilde{E}_{0,\nu,J}\,\tilde{\phi}_{0,\nu,J,m_J}(\vec{R}) \rightarrow$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2\tilde{R}_{0,\nu,J}}{dR^2} + \frac{\hbar^2 J(J+1)}{2\mu R^2}\tilde{R}_{0,\nu,J} + E_0(R)\,\tilde{R}_{0,\nu,J} = \tilde{E}_{0,\nu,J}\,\tilde{R}_{0,\nu,J}$$

Oso zaila garatzea (H atomoaren kasuan $E_0(R)$-k adierazpen coulombiarra izanez erraza izan zen $\Rightarrow$ emaitza analitikoa) $\Rightarrow$ HURBILKETAK

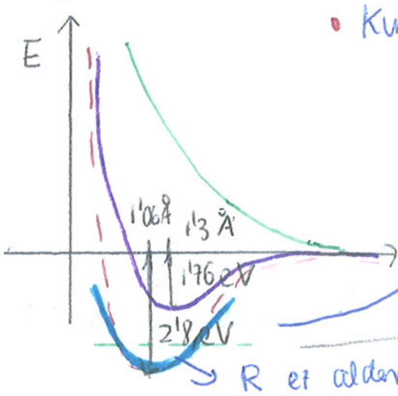
* $\frac{\hbar^2 J(J+1)}{2\mu R^2} \rightarrow$ nukleoak ez dira asko mugitzen: $R \sim R_0$

(ez da asko aldentzen $R_0$-tik)

R-ren balioa non $E_0(R)$ minimoa den $\rightarrow$ molekularen oreka distantzia orekan

$$\frac{\hbar^2 J(J+1)}{2\mu R^2} \simeq \frac{\hbar^2 J(J+1)}{2\mu R_0^2} = E_r \text{ (errotazioa)}$$

* $E_0(R) \Rightarrow H_2$-n adibidez:




R ez aldendu gehiegi $R_0$-n baliotik (pisutsuak direlako)

- Kurba parabolarekin ordezkatu $\Rightarrow$ bigarren ordeneko Taylor-en garapena $R_0$-ren inguruan (hurbilketa harmonikoa)
- Hurbilketaren egokitasuna energiaren araberakoa izango da.

$\rightarrow$ hasieran egokia $\rightarrow$ gero ez hainbeste (efektu anharmonikoa)

$$E_0(R) = E_0(R_0) + \frac{1}{2} \underbrace{\left.\frac{d^2E_0(R)}{dR^2}\right|_{R=R_0}}_{K} (R-R_0)^2 + O^3$$

$R_0$-n minimoa

duenez $\frac{dE_0}{dR}(R_0)=0$

Beraz $\Rightarrow$

$$-\frac{\hbar^2}{2\mu}\frac{d^2\tilde{R}_{0,\nu,J}}{dR^2} + \frac{1}{2}K(R-R_0)^2\tilde{R}_{0,\nu,J} = \underbrace{\left(\tilde{E}_{0,\nu,J} - E_r - E_0(R_0)\right)}_{=E'}\tilde{R}_{0,\nu,J}$$

Aldagai aldaketa $\Rightarrow$ $R' = R - R_0$ ; $dR' = dR$

$$-\frac{\hbar^2}{2\mu}\frac{d^2\tilde{R}_{0,\nu,J}}{dR'^2} + \frac{1}{2}KR'^2\tilde{R}_{0,\nu,J} = E'\tilde{R}_{0,\nu,J}$$ $\Rightarrow$ dimentsio bakarreko osziladore harmonikoa

$\omega_0 = \sqrt{\frac{K}{\mu}}$

$$\tilde{R}_{0,\nu,J}(R') = A\, e^{-\frac{\mu\omega_0}{2\hbar}R'^2} H_\nu\left(\sqrt{\frac{\mu\omega_0}{\hbar}}R'\right)$$

$\nu$ ordenako Hermiteren polinomioak

$$E' = \left(\frac{1}{2}+\nu\right)\hbar\omega_0$$

Aldagai aldaketa desegitez $\Rightarrow$ molekularen uhin-funtzioaren adierazpena:

- $$\tilde{\phi}_{0,J,m_J,\nu}(\vec{R}) = \frac{A}{R} H_\nu\left(\sqrt{\frac{\mu\omega_0}{\hbar}}(R-R_0)\right) e^{-\frac{\mu\omega_0}{2\hbar}(R-R_0)^2} Y_J^{m_J}(\theta,\varphi)$$

- $$\tilde{E}_{0,J,m_J} = E_r + E_0(R_0) + E' = \frac{\hbar^2 J(J+1)}{2I_0} + \left(\frac{1}{2}+\nu\right)\hbar\omega_0 + E_0(R_0)$$

errotazioaren lotutako ekarpena — bibrazioaren lotutako ekarpena — egoera elektronikoaren lotutako ekarpena

## MOLEKULA DIATOMIKOAREN BIRAKETA, BIBRAZIO eta ENERGIA ELEKTRONIKOAK:

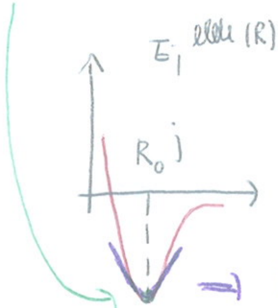
$$\tilde{E}_{j,\nu,J} = E_r^{bir} + E_\nu^{bib} + E_j^{elek}(R_0)$$

- $E_r^{bir} = \dfrac{\hbar^2 J(J+1)}{2 I_j}$ ; $I_j = \mu (R_0^{\,j})^2$  $J \in \mathbb{N}$

- $E_\nu^{bib} = \left(\dfrac{1}{2} + \nu\right) \hbar \omega_j$  $\nu \in \mathbb{N}$ , $\omega_j = \sqrt{\dfrac{K_j}{\mu}}$ , $K_j = \left(\dfrac{d^2 E_j(R)}{dR^2}\right)\Big|_{R=R_0^{\,j}}$

  inertzia-momentua

- $E_j^{elek}(R_0^{\,j})$  $e^-$-ak $j$ egoeran daudenean




$R_0^{\,j} \Rightarrow E_j^{elek}$ minimizatzen den $R$-ren posizioa

$\Rightarrow$ Kurbadura $\equiv K_j$

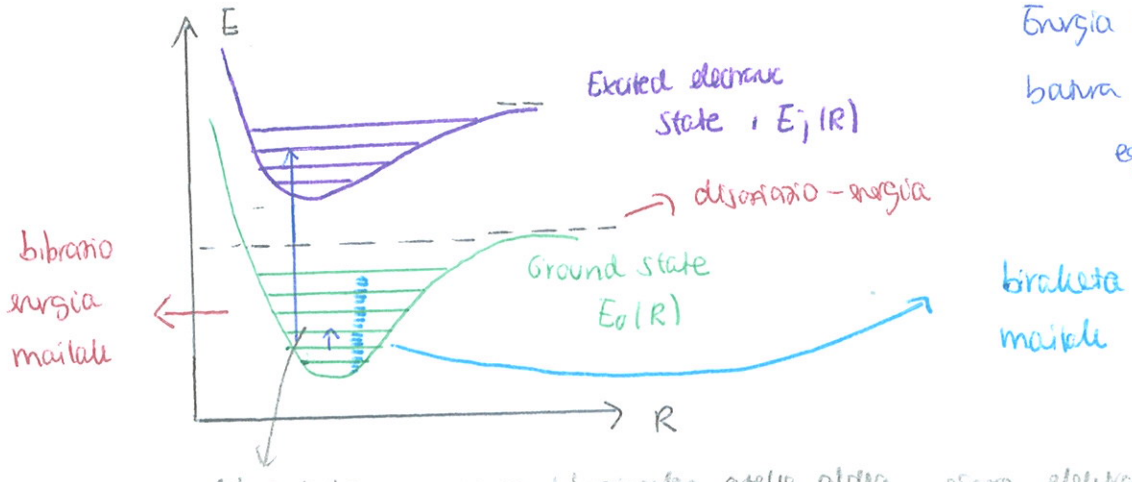
Energien ekarpen balioari dagokien magnitude-ordena oso ezberdina da.

* Adibidez, aldameneko $E_j^{elek}$-en aldea $\sim 1\,eV$ da. $\Rightarrow$ fotoien maiztasuna ikuskorra da (edo ultramorea)

* Bibrazioen energia txikiagoa da $\Rightarrow$ bi aldameneko bibrazio-mailen arteko aldea $\sim 0'1\,eV$-ekoa da $\Rightarrow$ fotoien maiztasuna infragorrikoa da

* Biraketen energia txikiena da $\Rightarrow$ $\simeq 0'001\,eV$ $\Rightarrow$ fotoiak mikrouhinekoak

Eskematikoki:




Energia osoa hiru hauen batura da $\Rightarrow$ ekarpen handiena egoera elektronikoarena

Asko txikiagoa egoera bibrazionalen arteko aldea egoera elektronikoen aldeak

Disoziazio-energia ⇒ disoziatzeko atomoen arteko distantzia infinitoa

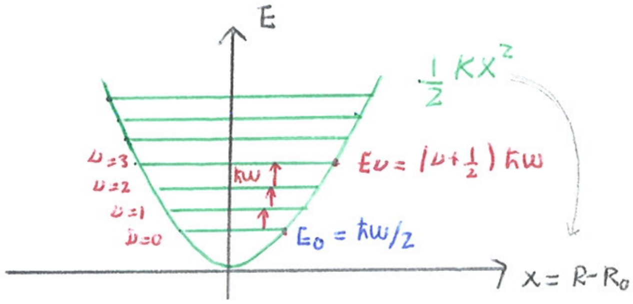
$$E_{disoziazio} = E_0(R \to \infty) - E_0(R = R_0)$$

putzuaren sakonera $E_0(R\to\infty)$-tik

Hau ekarpen elektronikoa kontuan hartuz soilik ⇒ biraketa eta bibrazio energia minimoak izan behar dira $R_0$-n ere. ⇒ biraketa energia minimoa $R_0$-n nulua da baina bibrazio-energia ez, $\frac{\hbar\omega_0}{2}$.

Beraz ⇒ $E_{disoziazio} = E_0(R\to\infty) - E_0(R=R_0) - \frac{1}{2}\hbar\omega_0$

* Elektroian egoera beste bat bada, $j$, $0$ beharrean $E_j(R\to\infty)$, $E_j(R=R_0)$ eta $\frac{\hbar\omega_j}{2}$ jarri.




## BIBRAZIO-ESPEKTROA:

- Bibrazioarekin lotutako energia-mailak:

$$\omega = \sqrt{\frac{K}{\mu}} \; ; \; E_\upsilon = \left(\upsilon + \frac{1}{2}\right)\hbar\omega$$

- Espektroa aztertzeko molekulen gainean eremu elektriko bat aplikatu behar dugu eta eremu horrekin lotutako fotoiak molekulak xurgatu ahal izango dira.
- Hala izateko, bibrazio-maila batetik bestera pasatu ahal izango dira ⇒ edozein trantsizio ez da posible:
- Eremu elektrikoak gure molekulekin elkarrekintza bat izango du:

(1. hurbilketa, momentu dipolaren hurbilketa): $\hat{H}_{elk} = -\hat{\vec{p}} \cdot \vec{E}(t)$

momentu dipolara — aplikatutako eremua

Denboraren menpeko perturbazio-teoria aplikatuz hasierako eta amaierako bibrazio-mailek hauxe bete behar dute (hautaketa-araua):

$$\Delta v = \pm 1 = v_j - v_i$$

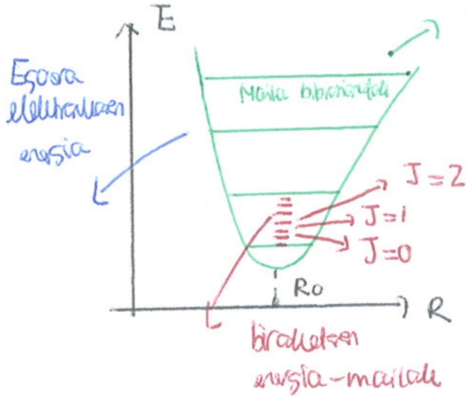
Molekula $v$ maila batean egonda aldamenekora pasatu ahal izango da bakarrik. ⇒ trantsizio horren energia aldaketa: $\Delta E = \pm \hbar\omega$

Fotoien maiztasuna ⇒ $\omega_{fotoi} = \omega$ (Fotoien maiztasuna molekularen maiztasunarekin bat datorrenean emango da soilik)

Beraz, bibrazio-espektroa aztertzen badugu molekularekin loturiko maiztasuna kalkulatu ahal izango dugu. Gainera, molekula ezaguna bada $\mu$ ezaguna dugu eta $k$ kalkulatu ahal izango dugu. ⇒ Molekulen erresistentziak ordenatu daitezke bibrazio-espektroa aztertuz

Nukleoen arteko indarra erresistentzia

## BIRAKETA-ESPEKTROA:




Molekularen egoera elektronikoa

$$E_J^{bir} = \frac{\hbar^2 J(J+1)}{2I_j} \qquad (I_j = \mu(R_0^j)^2)$$

- Biraketa-mailen arteko aldea oso txikia ⇒ ~ meV
- Eremu-elektriko bat aplikatzen badugu molekularen gainean, horrekin loturiko fotoiak izango ditugu eta molekulak horiek xurgatu edo igorri ahal izango ditu
- Biraketa mailen arteko trantsizioak egin ahal izango dira ⇒ edozein trantsizio ez da posible (arau batzuk)

• Kurbiltzeta dipolarra kontsideratzen badugu orain: $\hat{H}_{elk} = -\hat{\vec{p}} \cdot \vec{E}(t)$

$$\Delta J = \pm 1 \qquad \text{(hautapen-araua)}$$

Molekula aldamenetako mailetara bakarrik joan ahal izango da. ⇒

trantsizio horren energia aldaketa: $\Delta E = \frac{\hbar^2}{I}(J+1)$

energia diferentzia hau faktor mailaketak aztertu.

• Espektroa aztertuz molekularen inertzia kalkulatu daiteke (dagokion J ezaguna bada) ⇒ Gainera, molekula mota ezagutuz $\mu$ ezaguna da eta $R_0^J$ kalkulatu ahal izango dugu. (Nukleoen arteko oreka-distantzia)

↓ ordenaren zehaztapena dugu ezagutza elektronikoa onargarria dela

* Molekula kobalenteen oreka distantziak ionikoenak baino txikiagoak dira!

# 5. Ariketa orria

## FISIKA KUANTIKOA

17-03-16

1) Izan bitez $H = H_0 + \lambda\sqrt{\hbar m\omega^3}\,x$ , $H = H_0 + \lambda\cdot\frac{1}{2}m\omega^2 x$ eta $H = H_0 + \lambda\sqrt{\frac{m^3\omega^5}{\hbar}}\,x^3$

hamiltondarrak, $H_0$ dimentsio bakarreko osziladore harmonikoaren hamiltondarra izanik.

a) $H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$ , $H_0|\varphi_n^0\rangle = E_n|\varphi_n^0\rangle$ , $E_n^0 = (n+\frac{1}{2})\hbar\omega$ $n\in\mathbb{N}$

* $H = H_0 + \lambda\underbrace{\sqrt{\hbar m\omega^3}\,x}_{\tilde{\omega}}$ : $E_n(\lambda) = \varepsilon_0 + \lambda\varepsilon_1 + \lambda^2\varepsilon_2 + \dots$ , $|\psi(\lambda)\rangle = |0\rangle + \lambda|1\rangle + \lambda^2|2\rangle + \dots$

$\varepsilon_0 = E_n^0 = (n+\frac{1}{2})\hbar\omega$ , $|0\rangle = |\varphi_n^0\rangle$

$$\varepsilon_1 = \langle\varphi_n^0|\tilde{\omega}|\varphi_n^0\rangle = \sqrt{\hbar m\omega^3}\,\langle\varphi_n^0|x|\varphi_n^0\rangle = \sqrt{\hbar m\omega^3}\,\langle\varphi_n^0|\left(\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)\right)|\varphi_n^0\rangle =$$

$$\hbar\sqrt{\frac{m\omega^3}{2m\omega}}\,\langle\varphi_n^0|(\hat{a}+\hat{a}^+)|\varphi_n^0\rangle = \frac{\hbar\omega}{\sqrt{2}}\left[\langle\varphi_n^0|\hat{a}|\varphi_n^0\rangle + \langle\varphi_n^0|\hat{a}^+|\varphi_n^0\rangle\right] =$$

ortonormalak

$$\frac{\hbar\omega}{\sqrt{2}}\left[\sqrt{n}\langle\varphi_n^0|\varphi_{n-1}^0\rangle + \sqrt{n+1}\langle\varphi_n^0|\varphi_{n+1}^0\rangle\right] = 0$$

$$\varepsilon_2 = \sum_{m\neq n}'\sum_i \frac{|\langle\varphi_m^{0}|\tilde{\omega}|\varphi_n^0\rangle|^2}{E_n^0 - E_m^0} = \sum_{m\neq n}' \frac{|\langle\varphi_m^0|\sqrt{\hbar m\omega^3}\,x|\varphi_n^0\rangle|^2}{\hbar\omega(n-m)} = \frac{\hbar m\omega^3}{\hbar\omega}\cdot$$

$$\sum_{m\neq n}' \frac{|\langle\varphi_m^0|x|\varphi_n^0\rangle|^2}{(n-m)} = m\omega^2\sum_{m\neq n}' \frac{|\langle\varphi_m^0|\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)|\varphi_n^0\rangle|^2}{(n-m)} = \frac{m\omega^2\hbar}{2m\omega}\cdot$$

$$\sum_{m\neq n}' \frac{|\langle\varphi_m^0|(\hat{a}+\hat{a}^+)|\varphi_n^0\rangle|^2}{(n-m)} = \frac{\omega\hbar}{2}\sum_{m\neq n}' \frac{|\langle\varphi_m^0|\sqrt{n}\,\varphi_{n-1}^0\rangle + \langle\varphi_m^0|\sqrt{n+1}\,\varphi_{n+1}^0\rangle|^2}{(n-m)} =$$

$$\frac{\omega\hbar}{2}\sum_{m\neq n}' \frac{|\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}|^2}{(n-m)} = \frac{\omega\hbar}{2}\left[\frac{(\sqrt{n})^2}{n-(n-1)} + \frac{(\sqrt{n+1})^2}{n-(n+1)}\right] = +\frac{\omega\hbar}{2}\left(\frac{n}{1} + \frac{n+1}{-1}\right) =$$

$$-\frac{\omega\hbar}{2} \Rightarrow E_n(\lambda) = (n+\tfrac{1}{2})\hbar\omega - \lambda^2\frac{\omega\hbar}{2} = \omega\hbar\left(n+\frac{1}{2}-\frac{\lambda^2}{2}\right)$$

* $H = H_0 + \lambda \underbrace{\cdot \frac{1}{2} m\omega^2 x^2}_{\tilde{\omega}}$ : $E_n(\lambda) = \varepsilon_0 + \lambda \varepsilon_1 + \lambda^2 \varepsilon_2 + \dots$ , $|\psi(\lambda)\rangle = |0\rangle + \lambda |1\rangle + \lambda^2 |2\rangle + \dots$

$\varepsilon_0 = E_n^0 = (n+\frac{1}{2})\hbar\omega \Rightarrow |0\rangle = |\varphi_n^0\rangle$

$\varepsilon_1 = \langle \varphi_n^0 | \tilde{\omega} | \varphi_n^0 \rangle = \langle \varphi_n^0 | (\frac{1}{2} m\omega^2 x^2) | \varphi_n^0 \rangle = \frac{m\omega^2}{2} \langle \varphi_n^0 | x^2 | \varphi_n^0 \rangle = \frac{m\omega^2}{2} \cdot \frac{\hbar}{2m\omega}$

$\langle \varphi_n^0 | (\hat{a} + \hat{a}^+)^2 | \varphi_n^0 \rangle = \frac{\hbar\omega}{4} \langle \varphi_n^0 | (\hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}) | \varphi_n^0 \rangle = \frac{\hbar\omega}{4} [ \underbrace{\langle \varphi_n^0 | \hat{a}^2 | \varphi_n^0 \rangle}_{0} +$

$\underbrace{\langle \varphi_n^0 | \hat{a}^{+2} | \varphi_n^0 \rangle}_{0} + \langle \varphi_n^0 | \hat{a}\hat{a}^+ | \varphi_n^0 \rangle + \langle \varphi_n^0 | \hat{a}^+\hat{a} | \varphi_n^0 \rangle ] = \frac{\hbar\omega}{4} [ \langle \varphi_n^0 | ((n+1)\varphi_n^0) \rangle + \langle \varphi_n^0 | n\varphi_n^0 \rangle ] =$

$\frac{\hbar\omega}{4} [n+1+n] = (2n+1)\frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}(n+\frac{1}{2}) = \frac{\varepsilon_0}{2}$

$\varepsilon_2 = \sum_{m \neq n}' \sum_i \frac{|\langle \varphi_m^0 | \tilde{\omega} | \varphi_n^0 \rangle|^2}{E_n^0 - E_m^0} = \sum_{m\neq n}' \frac{|\langle \varphi_m^0 | (\frac{m\omega^2}{2} x^2) | \varphi_n^0 \rangle|^2}{(n-m)\hbar\omega} = \frac{m^2\omega^4}{\hbar\omega 4} \cdot$

$\sum_{m\neq n} \frac{|\langle \varphi_m^0 | x^2 | \varphi_n^0 \rangle|^2}{(n-m)} = \frac{\cancel{m^2}\omega^3}{4\cancel{\hbar}} \cdot \frac{\hbar^2}{4\cancel{m^2}\cancel{\omega^2}} \sum_{m\neq n}' \frac{|\langle \varphi_m^0 | (\hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}) | \varphi_n^0 \rangle|^2}{(n-m)} =$

$\frac{\hbar\omega}{16} \sum_{m\neq n}' \frac{|\sqrt{n(n-1)} \langle \varphi_m^0 | \varphi_{n-2}^0 \rangle + \sqrt{(n+1)(n+2)} \langle \varphi_m^0 | \varphi_{n+2}^0 \rangle + n \langle \varphi_m^0 | \varphi_n^0 \rangle + (n+1) \langle \varphi_m^0 | \varphi_n^0 \rangle|^2}{(n-m)} =$

$\to n \neq m$

$\frac{\hbar\omega}{16} \sum_{m\neq n}' \frac{|\sqrt{n(n-1)}\, \delta_{m,n-2} + \sqrt{(n+1)(n+2)}\, \delta_{m,n+2} + \cancel{n\, \delta_{m,n}} + \cancel{(n+1)\delta_{m,n}}|^2}{(n-m)} =$

$\frac{\hbar\omega}{16} \left[ \frac{n(n-1)}{n-(n-2)} + \frac{(n+1)(n+2)}{n-(n+2)} \right] = \frac{\hbar\omega}{16} \left[ \frac{n^2-n}{2} + \frac{(n+1)(n+2)}{-2} \right] = \frac{\hbar\omega}{16} \left( \frac{\cancel{n^2} - n - \cancel{n^2} - 3n - 2}{2} \right) =$

$\frac{\hbar\omega}{32}(-4n-2) = -\frac{4\hbar\omega}{32}(n+\frac{1}{2}) = -\frac{\hbar\omega}{8}(n+\frac{1}{2}) = -\frac{\varepsilon_0}{8}$

Orduan $\Rightarrow$ $E_n(\lambda) = (n+\frac{1}{2})\hbar\omega + \lambda \frac{\hbar\omega}{2}(n+\frac{1}{2}) - \lambda^2 \frac{\hbar\omega}{8}(n+\frac{1}{2}) = \hbar\omega(n+\frac{1}{2})(1+\frac{\lambda}{2} - \frac{\lambda^2}{8})$

* $H = H_0 + \lambda \underbrace{\sqrt{\frac{m^3\omega^5}{\hbar}}\, x^3}_{\tilde{\omega}}$ : $E_n(\lambda) = \varepsilon_0 + \lambda\varepsilon_1 + \lambda^2 \varepsilon_2 + \dots$ , $|\psi(\lambda)\rangle = |0\rangle + \lambda|1\rangle + \lambda^2 |2\rangle + \dots$

$$\varepsilon_0 = E_n^0 = \left(n+\tfrac{1}{2}\right)\hbar\omega \quad \Rightarrow \quad |0\rangle = |\varphi_n^0\rangle$$

$$\varepsilon_1 = \langle \varphi_n^0 | \hat{W} | \varphi_n^0 \rangle = \langle \varphi_n^0 | \left( \sqrt{\frac{m^3\omega^5}{\hbar}} x^3 \right) | \varphi_n^0 \rangle = \sqrt{\frac{m^3\omega^5}{\hbar}} \langle \varphi_n^0 | x^3 | \varphi_n^0 \rangle =$$

$$\sqrt{\frac{m^3\omega^5}{\hbar}} \langle \varphi_n^0 | \left( \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}+\hat{a}^+) \right)^3 | \varphi_n^0 \rangle = \sqrt{\frac{m^3\omega^5}{\hbar}} \sqrt{\frac{\hbar^3}{8m^3\omega^3}} \langle \varphi_n^0 | (\hat{a}+\hat{a}^+)^3 | \varphi_n^0 \rangle =$$

$$\frac{\hbar\omega}{2\sqrt{2}} \left[ \langle \varphi_n^0 | \left( \hat{a}^3 + \hat{a}^2\hat{a}^+ + \hat{a}^{+2}\hat{a} + \hat{a}^{+3} + \hat{a}\hat{a}^+\hat{a} + \hat{a}\hat{a}^{+2} + \hat{a}^+\hat{a}^2 + \hat{a}^+\hat{a}\hat{a}^+ \right) | \varphi_n^0 \rangle \right]$$

$$\frac{\hbar\omega}{2\sqrt{2}} \left[ \langle \varphi_n^0 | \cancelto{0}{\hat{a}^3} | \varphi_n^0 \rangle + \langle \varphi_n^0 | \cancelto{0}{\hat{a}^2\hat{a}^+} | \varphi_n^0 \rangle + \langle \varphi_n^0 | \cancelto{0}{\hat{a}^{+2}\hat{a}} | \varphi_n^0 \rangle + \langle \varphi_n^0 | \cancelto{0}{\hat{a}^{+3}} | \varphi_n^0 \rangle + \right.$$

$$\left. \langle \varphi_n^0 | \cancelto{0}{\hat{a}\hat{a}^+\hat{a}} | \varphi_n^0 \rangle + \langle \varphi_n^0 | \cancelto{0}{\hat{a}\hat{a}^{+2}} | \varphi_n^0 \rangle + \langle \varphi_n^0 | \cancelto{0}{\hat{a}^+\hat{a}^2} | \varphi_n^0 \rangle + \langle \varphi_n^0 | \cancelto{0}{\hat{a}^+\hat{a}\hat{a}^+} | \varphi_n^0 \rangle \right] =$$

$$0 \cdot \frac{\hbar\omega}{2\sqrt{2}} = 0$$

$$\varepsilon_2 = \sum_{m\neq n}{}' \sum_i \frac{|\langle \varphi_m^{0i} | \hat{W} | \varphi_n^0 \rangle|^2}{E_n^0 - E_m^0} = \sum_{m\neq n}{}' \frac{|\langle \varphi_m^0 | \sqrt{\frac{m^3\omega^5}{\hbar}} x^3 | \varphi_n^0 \rangle|^2}{(n-m)\hbar\omega} =$$

$$\frac{m^3\omega^5}{\hbar} \sum_{m\neq n}{}' \frac{|\langle \varphi_m^0 | \left( \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}+\hat{a}^+) \right)^3 | \varphi_n^0 \rangle|^2}{(n-m)\hbar\omega} = \frac{\cancel{m^3}\omega^{\cancel{5}}}{\cancel{\hbar}} \cdot \frac{\hbar^{\cancel{3}}}{8\cancel{m^3}\cancel{\omega^3}} \cdot \frac{1}{\cancel{\hbar\omega}} \cdot$$

$$\sum_{m\neq n}{}' \frac{|\langle \varphi_m^0 | (\hat{a}+\hat{a}^+)^3 | \varphi_n^0 \rangle|^2}{(n-m)} = \frac{\hbar\omega}{8} \sum_{m\neq n}{}' \frac{|\langle \varphi_m^0 | \hat{a}^3 | \varphi_n^0 \rangle + \langle \varphi_m^0 | \hat{a}^2\hat{a}^+ | \varphi_n^0 \rangle +}{(n-m)}$$

$$\frac{\langle \varphi_m^0 | \hat{a}^{+2}\hat{a} | \varphi_n^0 \rangle + \langle \varphi_m^0 | \hat{a}^{+3} | \varphi_n^0 \rangle + \langle \varphi_m^0 | \hat{a}\hat{a}^+\hat{a} | \varphi_n^0 \rangle + \langle \varphi_m^0 | \hat{a}\hat{a}^{+2} | \varphi_n^0 \rangle +}{(n-m)}$$

$$\frac{\langle \varphi_m^0 | \hat{a}^+\hat{a}^2 | \varphi_n^0 \rangle + \langle \varphi_m^0 | \hat{a}^+\hat{a}\hat{a}^+ | \varphi_n^0 \rangle|^2}{(n-m)} = \frac{\hbar\omega}{8} \sum_{m\neq n}{}' \frac{|\sqrt{n(n-1)(n-2)}\,\delta_{m,n-3} + \sqrt{n}(n+1)\,\delta_{m,n-1}}{(n-m)}$$

$$\frac{+\, n\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{(n+1)(n+2)(n+3)}\,\delta_{m,n+3} + n\sqrt{n}\,\delta_{m,n-1} + (n+2)\sqrt{n+1}\,\delta_{m,n+1} + (n+1)\sqrt{n}\,\delta_{m,n-1} +}{(n-m)}$$

$$\frac{(n+1)\sqrt{n+1}\,\delta_{m,n+1}|^2}{(n-m)} = \frac{\hbar\omega}{8} \left( \frac{n(n-1)(n-2)}{3} - \frac{(n+1)(n+2)(n+3)}{3} + 9n^3 - 9(n+1)^3 \right) =$$

$$\frac{\hbar\omega}{8}\left[\frac{-\cancel{3}(3n^2+3n+2)}{\cancel{3}} - 9(3n^2+3n+1)\right] = -\frac{\hbar\omega}{8}\left(3n^2+3n+2+27n^2+27n+9\right) =$$

$$-\frac{\hbar\omega}{8}\left(+30n^2+30n+11\right) \Rightarrow E_n(\lambda) = \hbar\omega\left(n+\frac{1}{2}\right) - \lambda^2\frac{\hbar\omega}{8}\left(30n^2+30n+11\right)$$

b) $|\varphi_n^0\rangle \Rightarrow$ osziladore harmonikoaren autobektoreak, $H_0|\varphi_n^0\rangle = E_n|\varphi_n^0\rangle$

* $H = H_0 + \lambda\underbrace{\sqrt{\hbar m\omega^3}\,x}_{\tilde{\omega}}$ $\qquad |\psi_n(\lambda)\rangle = |0\rangle + \lambda|1\rangle + \dots$

$\langle\varphi_m^0|1\rangle = \frac{1}{E_n^0-E_m^0}\langle\varphi_m^0|\tilde{\omega}|\varphi_n\rangle$

$|0\rangle = |\varphi_n^0\rangle$ (ez-endekatua da $E_n^0$)

$$|1\rangle = \sum_{m\neq n}{}' \sum_i \langle\varphi_m^{0\,i}|1\rangle|\varphi_m^{0\,i}\rangle = \sum_{m\neq n}{}' \langle\varphi_m^0|1\rangle|\varphi_m^0\rangle = \sum_{m\neq n}{}' \frac{\langle\varphi_m^0|\tilde{\omega}|\varphi_n^0\rangle}{E_n^0-E_m^0}|\varphi_m^0\rangle$$

- $\langle\varphi_m^0|\tilde{\omega}|\varphi_n^0\rangle = \langle\varphi_m^0|\sqrt{\hbar m\omega^3}\,x|\varphi_n^0\rangle = \sqrt{\hbar m\omega^3}\,\langle\varphi_m^0|x|\varphi_n^0\rangle =$

$$\sqrt{\hbar \cancel{m}\omega^{3/2}}\sqrt{\frac{\hbar}{2\cancel{m}\cancel{\omega}}}\,\langle\varphi_m^0|(\hat{a}+\hat{a}^+)|\varphi_n^0\rangle = \frac{\hbar\omega}{\sqrt{2}}\left[\langle\varphi_m^0|\hat{a}|\varphi_n^0\rangle + \langle\varphi_m^0|\hat{a}^+|\varphi_n^0\rangle\right] =$$

$$\frac{\hbar\omega}{\sqrt{2}}\left[\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right]$$

$$\Rightarrow |1\rangle = \sum_{m\neq n}{}' \frac{\frac{\cancel{\hbar\omega}}{\sqrt{2}}\left[\sqrt{n}\,\delta_{m,n-1}+\sqrt{n+1}\,\delta_{m,n+1}\right]}{\cancel{\hbar\omega}\,(n-m)}|\varphi_m^0\rangle = \frac{1}{\sqrt{2}}\left[\sqrt{n}\,|\varphi_{n-1}^0\rangle - \sqrt{n+1}\,|\varphi_{n+1}^0\rangle\right]$$

Orduan $\Rightarrow |\psi_n(\lambda)\rangle = |\varphi_n^0\rangle + \frac{\lambda}{\sqrt{2}}\left[\sqrt{n}\,|\varphi_{n-1}^0\rangle - \sqrt{n+1}\,|\varphi_{n+1}^0\rangle\right]$

* $H = H_0 + \lambda\cdot\underbrace{\frac{1}{2}m\omega^2x^2}_{\tilde{\omega}}$ $\qquad |\psi_n(\lambda)\rangle = |0\rangle + \lambda|1\rangle + \dots$

$|0\rangle = |\varphi_n^0\rangle$ (ez-endekatua da $E_n^0$)

$$|1\rangle = \sum_{m\neq n}{}' \frac{\langle\varphi_m^0|\tilde{\omega}|\varphi_n^0\rangle}{E_n^0-E_m^0}|\varphi_m^0\rangle$$

- $\langle\varphi_m^0|\tilde{\omega}|\varphi_n^0\rangle = \left(\frac{m\omega^2}{2}\right)\langle\varphi_m^0|x^2|\varphi_n^0\rangle = \frac{\cancel{m}\omega^{\cancel{2}}}{2}\frac{\hbar}{2\cancel{m}\cancel{\omega}}\langle\varphi_m^0|(\hat{a}^2+\hat{a}^{+2}+\hat{a}\hat{a}^++\hat{a}^+\hat{a})|\varphi_n^0\rangle =$

$$\frac{\hbar\omega}{4}\left[\sqrt{n(n-1)}\,\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + n\,\delta_{m,n} + (n+1)\,\delta_{m,n}\right]$$

$$\Rightarrow |1\rangle = \sum_{m\neq n}' \frac{\frac{\cancel{\hbar\omega}}{4}\left[\sqrt{n(n-1)}\,\delta_{m,n-2} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + n\,\delta_{m,n} + (n+1)\delta_{m,n}\right]}{\cancel{\hbar\omega}(n-m)} |\varphi_m^0\rangle =$$

$$\frac{1}{4}\left[\frac{\sqrt{n(n-1)}}{2}|\varphi_{n-2}^0\rangle - \frac{\sqrt{(n+1)(n+2)}}{2}|\varphi_{n+2}^0\rangle\right] = \frac{1}{8}\left[\sqrt{n(n-1)}\,|\varphi_{n-2}^0\rangle - \sqrt{(n+1)(n+2)}\,|\varphi_{n+2}^0\rangle\right]$$

Orduan, $|\varphi_n(\lambda)\rangle = |\varphi_n^0\rangle + \frac{\lambda}{8}\left[\sqrt{n(n-1)}\,|\varphi_{n-2}^0\rangle - \sqrt{(n+1)(n+2)}\,|\varphi_{n+2}^0\rangle\right]$

* $H = H_0 + \lambda \underbrace{\sqrt{\frac{m^3\omega^5}{\hbar}}\, x^3}_{\hat{\omega}}$  $|\varphi_n(\lambda)\rangle = |0\rangle + \lambda|1\rangle + \ldots$

$|0\rangle = |\varphi_n^0\rangle$ ; $|1\rangle = \sum_{m\neq n}' \frac{\langle\varphi_m^0|\hat{\omega}|\varphi_n^0\rangle}{E_n^0 - E_m^0}|\varphi_m^0\rangle$

- $\langle\varphi_m^0|\hat{\omega}|\varphi_n^0\rangle = \sqrt{\frac{m^3\omega^5}{\hbar}}\langle\varphi_m^0|x^3|\varphi_n^0\rangle = \sqrt{\frac{\cancel{m^3}\omega^5}{\hbar}}\sqrt{\frac{\hbar^3}{8\cancel{m^3}\omega^3}}\langle\varphi_m^0|(\hat{a}+\hat{a}^+)^3|\varphi_n^0\rangle =$

$$\frac{\hbar\omega}{\sqrt{8}}\left[\sqrt{n(n-1)(n-2)}\,\delta_{m,n-3} + 3n\sqrt{n}\,\delta_{m,n-1} + 3(n+1)\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{(n+1)(n+2)(n+3)}\,\delta_{m,n+3}\right]$$

$$\Rightarrow |1\rangle = \frac{\cancel{\hbar\omega}}{\sqrt{8}}\sum_{m\neq n}' \frac{\left[\sqrt{n(n-1)(n-2)}\,\delta_{m,n-3} + 3n\sqrt{n}\,\delta_{m,n-1} + 3(n+1)\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{(n+1)(n+2)(n+3)}\,\delta_{m,n+3}\right]}{\cancel{\hbar\omega}(n-m)}|\varphi_m^0\rangle$$

$$\frac{1}{\sqrt{8}}\left[\frac{\sqrt{n(n-1)(n-2)}}{3}|\varphi_{n-3}^0\rangle + 3n\sqrt{n}\,|\varphi_{n-1}^0\rangle - 3(n+1)\sqrt{n+1}\,|\varphi_{n+1}^0\rangle - \frac{\sqrt{(n+1)(n+2)(n+3)}}{3}|\varphi_{n+3}^0\rangle\right]$$

Orduan, $|\varphi_n(\lambda)\rangle = |\varphi_n^0\rangle + \frac{\lambda}{\sqrt{8}}\left[\frac{\sqrt{n(n-1)(n-2)}}{3}|\varphi_{n-3}^0\rangle - \frac{\sqrt{(n+1)(n+2)(n+3)}}{3}|\varphi_{n+3}^0\rangle + 3n\sqrt{n}\,|\varphi_{n-1}^0\rangle +\right.$

$\left. - 3n(n+1)\,|\varphi_{n+1}^0\rangle\right]$

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c) Adierazpen zehatza $\Rightarrow$ $H = H_0 + \lambda\sqrt{\hbar m\omega^3}\, x = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 + \lambda\sqrt{\hbar m\omega^3}\, x$

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}\left(x + \frac{\lambda\sqrt{\hbar m\omega^3}}{m\omega^2}\right)^2 - \frac{\hbar\cancel{m}\omega^{\cancel{3}}\lambda^2}{2\cancel{m\omega^2}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2} + \frac{m\omega^2}{2}(x')^2 - \frac{\hbar\omega}{2}\lambda^2$

$H|\psi_n\rangle = \mathcal{E}|\psi_n\rangle \Rightarrow \mathcal{E} = (n+\frac{1}{2})\hbar\omega - \frac{\lambda^2}{2}\hbar\omega$

$|\psi_n\rangle = e^{-\frac{\lambda}{\sqrt{2}}(\hat{a}^+-\hat{a})}|\varphi_n^0\rangle$ ( $x$ ordezen traslazioa : $e^{-ia\hat{p}/\hbar}$ traslazio eragilea

*'

a traslatatutako distantzia bada, $a = -\lambda\sqrt{\frac{\hbar m\omega^3}{m^2\omega^4}} = -\lambda\sqrt{\frac{\hbar}{m\omega}}$ )

$$|\psi_n\rangle = e^{-\frac{\lambda}{\sqrt{2}}(\hat{a}^+-\hat{a})}|\varphi_n^0\rangle \underset{\substack{\lambda\ll \\ (\text{Taylor})}}{\simeq} \left[1-\frac{\lambda}{\sqrt{2}}(\hat{a}^+-\hat{a})\right]|\varphi_n^0\rangle = |\varphi_n^0\rangle - \frac{\lambda}{\sqrt{2}}\sqrt{n+1}\,|\varphi_{n+1}^0\rangle + \frac{\lambda}{\sqrt{2}}\sqrt{n}\,|\varphi_{n-1}^0\rangle$$

Beraz emaitza berbera lortzen da

$\varphi_n^0(x)$-n $x$ beharrean $x'$ jarri

$$*^1\ \langle x|\psi_n\rangle = \psi_n(x) = \langle(x+\sqrt{\tfrac{\hbar}{m\omega}}\lambda)|\varphi_n^0\rangle = \varphi_n^0(x+\underbrace{\sqrt{\tfrac{\hbar}{m\omega}}\lambda}) \Rightarrow \text{Taylor-en garapena}$$

$$\varphi_n^0(x) + \sqrt{\tfrac{\hbar}{m\omega}}\lambda\,\varphi_n^{0\,\prime}(x) + \dots$$

$*^2$ Besteen kalkulua zehatza

2.1 Izan bedi eremu elektriko astatiko eta uniforme baten eraginpean kokaturiko dimentsio bakarreko osziladore harmoniko kargatua.

a) $H = H_0 - qE_0 x = H_0 + W$ , $\quad H_0|\varphi_n^0\rangle = E_n^0|\varphi_n^0\rangle$, $\quad E_n^0 = (n+\frac{1}{2})\hbar\omega \quad n\in\mathbb{N}$

$$|\psi(\lambda)\rangle = |0\rangle + \lambda|1\rangle + \dots$$

$$|0\rangle = |\varphi_n^0\rangle \quad , \quad \varepsilon_0 = E_n^0 \quad , \quad \lambda|1\rangle = \sum_{m\neq n}{}' \sum_i \frac{\langle\varphi_m^{0\,i}|W|\varphi_n^0\rangle}{E_n^0 - E_m^0}|\varphi_m^i\rangle =$$

$$\sum_{m\neq n}{}' \frac{\langle\varphi_m^0|W|\varphi_n^0\rangle}{E_n^0 - E_m^0}|\varphi_m^0\rangle$$

$$*\ \langle\varphi_m^0|W|\varphi_n^0\rangle = -qE_0\langle\varphi_m^0|x|\varphi_n^0\rangle = -qE_0\langle\varphi_m^0|\sqrt{\tfrac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)|\varphi_n^0\rangle =$$

$$-qE_0\sqrt{\frac{\hbar}{2m\omega}}\left[\langle\varphi_m^0|\hat{a}|\varphi_n^0\rangle + \langle\varphi_m^0|\hat{a}^+|\varphi_n^0\rangle\right] = -qE_0\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\langle\varphi_m^0|\varphi_{n-1}^0\rangle + \right.$$

$$\left.\sqrt{n+1}\,\langle\varphi_m^0|\varphi_{n+1}^0\rangle\right] = -qE_0\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right]$$

$$\lambda|1\rangle = \sum_{m\neq n}{}' \frac{-qE_0\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{m,n-1}+\sqrt{n+1}\,\delta_{m,n+1}\right]}{\hbar\omega(n-m)}|\varphi_m^0\rangle = -\frac{qE_0}{\omega}\sqrt{\frac{1}{2m\omega\hbar}}\left[\sqrt{n}\,|\varphi_{n-1}^0\rangle - \sqrt{n+1}\,|\varphi_{n+1}^0\rangle\right]$$

$$|\psi(\lambda)\rangle = |\varphi_n^0\rangle - \frac{qE_0}{\omega}\sqrt{\frac{1}{2m\omega\hbar}}\left[\sqrt{n}\,|\varphi_{n-1}^0\rangle - \sqrt{n+1}\,|\varphi_{n+1}^0\rangle\right]$$

$$*\ \langle P_x\rangle = q\langle\psi(\lambda)|x|\psi(\lambda)\rangle = q\langle\psi(\lambda)|\sqrt{\tfrac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)|\psi(\lambda)\rangle = q\sqrt{\tfrac{\hbar}{2m\omega}}\langle\psi(\lambda)|(\hat{a}+\hat{a}^+)|\psi(\lambda)\rangle =$$

$$q\sqrt{\frac{\hbar}{2m\omega}}\Big[\langle\varphi_n^0|\cancel{(\hat a+\hat a^+)|\varphi_n^0\rangle}^{0} - \frac{qE_0}{\omega}\sqrt{\frac{1}{2m\omega\hbar}}\sqrt{n}\langle\varphi_n^0|(\hat a+\hat a^+)|\varphi_{n-1}^0\rangle + \frac{qE_0}{\omega}\sqrt{\frac{1}{2m\omega\hbar}}\sqrt{n+1}$$

$$\langle\varphi_n^0|(\hat a+\hat a^+)|\varphi_{n+1}^0\rangle - \frac{qE_0}{\omega}\sqrt{\frac{n}{2m\omega\hbar}}\langle\varphi_{n-1}^0|(\hat a+\hat a^+)|\varphi_n^0\rangle + \left(\frac{qE_0}{\omega}\right)^2\frac{\sqrt{n(n+1)}}{2m\omega\hbar}\langle\varphi_{n-1}^0\cancel{|(\hat a+\hat a^+)|\varphi_{n+1}^0\rangle}^{0} +$$

$$\frac{qE_0}{\omega}\sqrt{\frac{n+1}{2m\omega\hbar}}\langle\varphi_{n+1}^0|(\hat a+\hat a^+)|\varphi_n^0\rangle + \left(\frac{qE_0}{\omega}\right)^2\frac{\sqrt{n(n+1)}}{2m\omega\hbar}\langle\varphi_{n+1}^0\cancel{|(\hat a+\hat a^+)|\varphi_{n-1}^0\rangle}^{0}\Big] =$$

$$q\sqrt{\frac{\hbar}{2m\omega}}\left[-\frac{qE_0}{\omega}\sqrt{\frac{n}{2m\omega\hbar}}\sqrt{n} + \frac{qE_0}{\omega}\sqrt{\frac{n+1}{2m\omega\hbar}}\sqrt{n+1} - \frac{qE_0}{\omega}\sqrt{\frac{n}{2m\omega\hbar}}\sqrt{n} + \frac{qE_0}{\omega}\sqrt{\frac{n+1}{2m\omega\hbar}}\sqrt{n+1}\right] =$$

$$q^2\frac{E_0}{\omega}\sqrt{\frac{\cancel{\hbar}}{2m\omega}}\frac{1}{\sqrt{2m\omega\cancel{\hbar}}}\left[-\cancel{n}+(\cancel{n}+1)-\cancel{n}+(\cancel{n}+1)\right] = \cancel{2}q^2\frac{E_0}{\omega}\frac{1}{\cancel{2}m\omega} = \underbrace{\frac{q^2}{m\omega^2}}_{\alpha}E_0$$

b) $H = H_0 - qE_0x = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}Kx^2 - qE_0x = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{q^2E_0^2}{2K} + \frac{1}{2}K\left(x-\frac{qE_0}{K}\right)^2 =$

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2} - \frac{q^2E_0^2}{2K} + \frac{1}{2}K(x')^2$ ; $\varepsilon_n' = \varepsilon - \frac{q^2E_0^2}{2K}$ $\quad\left(x' = x - \frac{qE_0}{K}\ ;\ \omega = \sqrt{\frac{K}{m}}\right)$

Hamiltondarra desplazatuta x ardatzean $a = \frac{qE_0}{K}$ distantzian:

$$|\varphi_n'\rangle = e^{-i\frac{qE_0}{m\omega^2}\frac{\hat p}{\hbar}}|\varphi_n^0\rangle = e^{-i\frac{qE_0}{m\omega^2\hbar}\left(-i\sqrt{\frac{\hbar m\omega}{2}}(\hat a-\hat a^+)\right)}|\varphi_n^0\rangle =$$

$$e^{-\frac{qE_0}{\omega}\frac{(\hat a-\hat a^+)}{\sqrt{2m\hbar\omega}}}|\varphi_n^0\rangle \simeq \left[1-\frac{qE_0}{\omega}\frac{(\hat a-\hat a^+)}{\sqrt{2m\hbar\omega}}\right]|\varphi_n^0\rangle = |\varphi_n^0\rangle - \frac{qE_0}{\omega}\sqrt{\frac{n}{2m\hbar\omega}}|\varphi_{n-1}^0\rangle +$$

$$+\frac{qE_0}{\omega}\sqrt{\frac{n+1}{2m\hbar\omega}}|\varphi_{n+1}^0\rangle$$

Emaitza bera

$\langle\psi_n|p_x|\psi_n\rangle = q\langle\psi_n|x|\psi_n\rangle =$
$q\langle\psi_n|x-\frac{qE}{m\omega^2}|\psi_n\rangle + \frac{q^2E}{m\omega^2} = q\langle n|x|n\rangle + \frac{q^2E}{m\omega^2}$

Beraz $\langle p_x\rangle$ egitean emaitza bera eta polarizagarritasun bera lortzen dugu.

3.) Izan bedi $H = H_0 + \lambda\hat W$ hamiltondarra, $H_0$ eta $\hat W$ ragileak honako hauek izanik: $H_0 = \begin{pmatrix} a & 0 \\ 0 & -a\end{pmatrix}$ eta $\hat W = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$

a) H hamiltondarraren autobalioteen garapena, λ parametroarekiko bigarren ordenako eta ordena altuagoko gaiak arbuiatuz.

Lehenengo $H_0$-ren autobektore eta autobalioak lortuko ditugu:

Suposatuz $\{|\varphi_1\rangle, |\varphi_2\rangle\}$ oinarrian garatuta daudela matrizeak.

$$|H_0 - E^0 \mathbb{1}| = \begin{vmatrix} a-E^0 & 0 \\ 0 & -a-E^0 \end{vmatrix} = (a-E^0)(-a-E^0) = 0 \rightarrow E^0_1 = a, \; E^0_2 = -a$$

- $E^0_1 = a \Rightarrow$ $$\begin{pmatrix} 0 & 0 \\ 0 & -2a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -2ay \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = 0 \quad |\psi_1^0\rangle = |\varphi_1\rangle$$

- $E^0_2 = -a \Rightarrow$ $$\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2ax \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = 0 \quad |\psi_2^0\rangle = |\varphi_2\rangle$$

Bestalde $H$-ren autobektoreen garapena $\lambda$-rekiko hauxe izango da.

$$|\psi(\lambda)\rangle = |0\rangle + \lambda|1\rangle + \dots$$

Bi autobektore izango ditu: $|\psi_1(\lambda)\rangle$ eta $|\psi_2(\lambda)\rangle$

- $|\psi_1(\lambda)\rangle = |0\rangle_1 + \lambda|1\rangle_1 + \dots \quad , \quad E_1(\lambda) = \varepsilon_0 + \lambda\varepsilon_1 + \lambda^2\varepsilon_2 + \dots$

$$\varepsilon_0 = E_1^0 \quad ; \quad |0\rangle_1 = |\psi_1^0\rangle \quad ; \quad |1\rangle_1 = \sum_{m\neq n}{}' \sum_i \frac{\langle\psi_{m}^{0i}|\tilde{W}|\psi_n^0\rangle}{E_n^0 - E_m^0}|\psi_{m i}^0\rangle =$$

$$\frac{\langle\psi_2^0|\tilde{W}|\psi_1^0\rangle}{E_1^0 - E_2^0}|\psi_2^0\rangle \overset{*}{=} \frac{|\psi_2^0\rangle}{2a}$$

* $|\varphi_1\rangle = |\psi_1^0\rangle$ eta $|\psi_2^0\rangle = |\varphi_2\rangle$ direnez $\tilde{W}$ $H_0$-ren autobektoreen

oinarrian garatuta dago eta beraz matrizetik $\langle\psi_2^0|\tilde{W}|\psi_1^0\rangle$ kalkulatu

dezakegu $\langle\psi_2^0|\tilde{W}|\psi_1^0\rangle = 1$

Beraz, $$|\psi_1(\lambda)\rangle = |\psi_1^0\rangle + \frac{\lambda}{2a}|\psi_2^0\rangle + O(\lambda^2)$$

- $|\psi_2(\lambda)\rangle = |0\rangle_2 + \lambda|1\rangle_2 + \dots \quad , \quad E_2(\lambda) = \varepsilon_0 + \lambda\varepsilon_1 + \lambda^2\varepsilon_2 + \dots$

$$\varepsilon_0 = E_2^0 \quad ; \quad |0\rangle_2 = |\psi_2^0\rangle \quad ; \quad |1\rangle_2 = \sum_{m\neq n}{}' \sum_i \frac{\langle\psi_{m}^{0i}|\tilde{W}|\psi_n^0\rangle}{E_n^0 - E_m^0}|\psi_{m i}^0\rangle =$$

$$\frac{\langle \psi_1^0 | \hat{W} | \psi_2^0 \rangle}{E_2^0 - E_1^0} |\psi_1^0\rangle \overset{*}{=} \frac{|\psi_1^0\rangle}{-2a} = -\frac{1}{2a} |\psi_1^0\rangle$$

* Berriro ere $\langle \psi_1^0 | \hat{W} | \psi_2^0 \rangle$ $\hat{W}$ matrizetik lortuko dugu: $\langle \psi_1^0 | \hat{W} | \psi_2^0 \rangle = 1$

Beraz,
$$|\psi_2(\lambda)\rangle = |\psi_2^0\rangle - \frac{\lambda}{2a} |\psi_1^0\rangle$$

b) Lor bitez H hamiltondarraren autobektoreen adierazpen zehatza, egin bedi beraren $\lambda$ parametroarekiko garapena, bigarren ordenako eta ordena altuagoko gaiak arbuiatuz, eta konpara bedi garapen hori (a) atalean lortutako emaitzarekin.

$$H = H_0 + \lambda \hat{W} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & \lambda \\ \lambda & -a \end{pmatrix}$$

Ho-ren autobektoreen oinarrian garatuta egongo da $\{|\psi_1^0\rangle, |\psi_2^0\rangle\}$

Bere autobektore eta autobalioak kalkulatuko ditugu:

$$|H - E\mathbb{1}| = \begin{vmatrix} a-E & \lambda \\ \lambda & -a-E \end{vmatrix} = (a-E)(-a-E) - \lambda^2 = -(a-E)(a+E) - \lambda^2 = 0 \rightarrow$$

$$-(a^2 - E^2) - \lambda^2 = E^2 - a^2 - \lambda^2 = E^2 - (a^2+\lambda^2) = 0 \rightarrow E_1 = \sqrt{a^2+\lambda^2}, \quad E_2 = -\sqrt{a^2+\lambda^2}$$

- $E_1 = \sqrt{a^2+\lambda^2} \Rightarrow$
$$\begin{pmatrix} a - \sqrt{a^2+\lambda^2} & \lambda \\ \lambda & -a-\sqrt{a^2+\lambda^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(a-\sqrt{a^2+\lambda^2}) + \lambda y \\ \lambda x - (a+\sqrt{a^2+\lambda^2}) y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\lambda x = (a + \sqrt{a^2+\lambda^2}) y \rightarrow x = \frac{(a+\sqrt{a^2+\lambda^2})}{\lambda} y = \frac{2a}{\lambda} y$$

$$\frac{\sqrt{a^2+\lambda^2}}{\lambda} = \frac{a}{\lambda}\sqrt{1 + \left(\frac{\lambda}{a}\right)^2} \underset{\lambda/a \ll 1}{\simeq} \frac{a}{\lambda}\left(1 + \frac{\lambda^2}{2a^2} - \frac{\lambda^4}{8a^4} + \cdots\right) \simeq \frac{a}{\lambda}$$

Orduan normalizatuz: $A^2\left(\frac{4a^2}{\lambda^2} + 1\right) = 1 \rightarrow A = \frac{1}{\sqrt{1+4\left(\frac{a}{\lambda}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{2a}{\lambda}\right)^2}}$

$$|\psi_1\rangle = \frac{1}{\sqrt{1+\left(\frac{2a}{\lambda}\right)^2}}\left(\frac{2a}{\lambda}|\psi_1^0\rangle + |\psi_2^0\rangle\right) = \frac{2a/\lambda}{\sqrt{1+\left(\frac{2a}{\lambda}\right)^2}}|\psi_1^0\rangle + \frac{1}{\sqrt{1+\left(\frac{2a}{\lambda}\right)^2}}|\psi_2^0\rangle =$$

$$\frac{1}{\sqrt{1+\left(\frac{\lambda}{2a}\right)^2}}|\psi_1^0\rangle + \frac{\lambda/2a}{\sqrt{1+\left(\frac{\lambda}{2a}\right)^2}}|\psi_2^0\rangle \underset{\lambda/2a \ll 1}{\simeq} |\psi_1^0\rangle + \frac{\lambda}{2a}|\psi_2^0\rangle$$

Emaitza bera.

- $E_2 = -\sqrt{a^2+\lambda^2} \Rightarrow \begin{pmatrix} a+\sqrt{a^2+\lambda^2} & \lambda \\ \lambda & -a+\sqrt{a^2+\lambda^2}\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix}(a+\sqrt{a^2+\lambda^2})x+\lambda y \\ \lambda x + (\sqrt{a^2+\lambda^2}-a)y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$-\frac{a+\sqrt{a^2+\lambda^2}}{\lambda}x = y = -x\left(\frac{a}{\lambda}+\frac{a}{\lambda}\sqrt{1+\left(\frac{\lambda}{a}\right)^2}\right) \underset{\lambda/a \ll 1}{\simeq} -x\left(\frac{a}{\lambda}+\frac{a}{\lambda}\right) = -\frac{2a}{\lambda}x$$

Normalizatuz: $\left(\frac{4a^2}{\lambda^2}+1\right)A^2 = 1 \rightarrow A = \frac{1}{\sqrt{1+\frac{4a^2}{\lambda^2}}} = \frac{1}{\sqrt{1+\left(\frac{2a}{\lambda}\right)^2}} = \frac{1}{\frac{2a}{\lambda}\sqrt{\left(\frac{\lambda}{2a}\right)^2+1}} = \frac{\lambda/2a}{\sqrt{1+(\lambda/2a)^2}}$

$$|\psi_2\rangle = \frac{\lambda/2a}{\sqrt{1+(\lambda/2a)^2}}\left(-|\psi_1^0\rangle + \frac{2a}{\lambda}|\psi_2^0\rangle\right) = \frac{1}{\sqrt{1+(\lambda/2a)^2}}|\psi_2^0\rangle - \frac{\lambda/2a}{\sqrt{1+(\lambda/2a)^2}}|\psi_1^0\rangle \underset{\lambda/2a \ll 1}{\simeq}$$

$$|\psi_2^0\rangle - \frac{\lambda}{2a}|\psi_1^0\rangle$$

Emaitza bera.

4.1 Izan bedi spin egoeren espazioan eragiten duen $H = H_0 + \lambda\hat{W}$ hamiltondarra, $H_0 = AS^2 + BS_z$ eta $\hat{W} = w_0 S_y$ izanik, $S = 1/2$ delarik. Lor bitez,

a) A, B eta $w_0$ dimentsioen unitateak.

$$[H] = J = [AS^2] = [A][S^2] = [A]\frac{Kg^2\,m^4}{s^2} \Rightarrow [A] = \frac{J\,s^2}{Kg^2 m^4} = \frac{\cancel{Kg}\,\cancel{m^2}/\cancel{s^2}\,\cancel{s^2}}{Kg^{\cancel{2}}\,m^{\cancel{4}}} =$$

$$\frac{1}{Kg\,m^2}\ ; \quad [H] = J = [BS_z] = [B][S_z] \Rightarrow [B] = \frac{[H]}{[S_z]} = \frac{J\,s}{Kg\,m^2} = \frac{\cancel{Kg\,m^2}/s^{\cancel{2}}\,\cancel{s}}{\cancel{Kg\,m^2}} = \frac{1}{s}$$

$\lambda$ adimentsionala da

$$[\hat{W}] = [w_0][S_y] = \frac{[H]}{[\lambda]} = [H] = J \rightarrow [w_0] = \frac{J}{[S_y]} = \frac{J\cdot s}{Kg\,m^2} = \frac{1}{s}$$

b) $H_0$ hamiltondarraren espektroa.

$$H_0|\varphi_n^0\rangle = (AS^2+BS_z)|\varphi_n^0\rangle = E_n^0|\varphi_n^0\rangle \Rightarrow |\varphi_n^0\rangle$$ $S^2$ eta $S_z$-ren

aldi bereko autobektoreak dira, $s=1/2$ denean: $\{|-\rangle, |+\rangle\}$

$$E_n^0 = A\,s(s+1)\hbar^2 + B\,m_s\hbar \underset{s=1/2}{=} \frac{3}{4}\hbar^2 A \pm \frac{B\hbar}{2} \qquad (m_s = \pm\frac{1}{2})$$

$$E_1^0 = \frac{3}{4}\hbar^2 A - \frac{B}{2}\hbar = \frac{\hbar}{2}\left(\frac{3}{2}\hbar A - B\right), \quad E_2^0 = \frac{3}{4}\hbar^2 A + \frac{B}{2}\hbar = \frac{\hbar}{2}\left(\frac{3}{2}\hbar A + B\right)$$

c) H hamiltondarraren autobalioen garapena, $\lambda$ parametroarekiko lehenengo ordenako eta ordena altuagoko gaiak arbuiatuz, eta autobektoreen garapena, $\lambda$ parametroarekiko bigarren ordenako eta ordena altuagoko gaiak arbuiatuz.

Lehendabizi $(S_y)$ kalkulatuko dugu $\{S^2, S_z\}$-ren aldiberako autobektoreen oinarrian $S=1/2$ denean: $\{|+\rangle, |-\rangle\}$: $(S_y) = i\frac{\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Orduan, $w_0 S_y = \tilde{W} = w_0 \, i\frac{\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Autobalioak: $E(\lambda) = \varepsilon_0 + \lambda\varepsilon_1 + \frac{\lambda^2}{2}\varepsilon_2 + \dots$ Autobektoreak: $|\Psi(\lambda)\rangle = |0\rangle + \lambda|1\rangle + \dots$

Bi autobektore eta autobalio izango ditu: $\{|\Psi_1(\lambda)\rangle, E_1(\lambda)\}$ eta $\{|\Psi_2(\lambda)\rangle, E_2(\lambda)\}$.

$\Psi_1(\lambda) \Rightarrow \quad E_1(\lambda) = E_1^0 = \frac{\hbar}{2}\left(\frac{3\hbar}{2}A - B\right), \quad |0\rangle_1 = |-\rangle$

Matrizeki

$$|1\rangle = \sum_{m\neq n}' \sum_i \frac{\langle\varphi_m^i|\tilde{W}|\varphi_n\rangle}{E_n^0 - E_m^0}|\varphi_m^i\rangle = \frac{\langle +|\tilde{W}|-\rangle}{-B\hbar}|+\rangle \overset{*}{=} \frac{-i w_0 \hbar}{2}\cdot\frac{1}{-B\hbar}|+\rangle = \frac{i|+\rangle}{2}\frac{w_0}{B}$$

Beraz $\Rightarrow \quad |\Psi_1(\lambda)\rangle = |-\rangle + \frac{i w_0}{2B}\lambda|+\rangle \rightarrow |\Psi_1(\lambda)\rangle = \frac{2B}{\sqrt{\lambda^2 w_0^2 + 4B^2}}\left(|-\rangle + \frac{i w_0}{2B}\lambda|+\rangle\right)$

$$\varepsilon_1 = \langle -|\hat{\omega}|-\rangle = 0 \quad , \quad \varepsilon_2 = \sum_{m\neq n}{}' \sum_{1} \frac{|\langle \varphi_m^1|\hat{\omega}|\varphi_n\rangle|^2}{E_n^0 - E_m^0} = \frac{1}{-B\hbar}|\langle +|\hat{\omega}|-\rangle|^2 \overset{*}{=}$$

* Matrizeak

$$-\frac{\omega_0^2\hbar^2}{4}\cdot\frac{1}{B\hbar} = -\frac{\omega_0^2\hbar}{4B} \Rightarrow E_1(\lambda) = E_1^0 - \frac{\omega_0^2\hbar}{4B}\lambda^2 = \frac{\hbar}{2}\left(\frac{3\hbar A}{2} - B\right) - \frac{\omega_0^2\hbar}{4B}\lambda^2$$

$\psi_2(\lambda) \Rightarrow$ $\quad E_2(\lambda) = E_2^0 = \frac{\hbar}{2}\left(\frac{3A\hbar}{2}+B\right), \quad |0\rangle = |+\rangle$

* Matrizeak.

$$|1\rangle = \sum_{m\neq n}{}'\sum_{1}\frac{\langle\varphi_m^1|\hat{\omega}|\varphi_n\rangle}{E_n^0 - E_m^0}|\varphi_m^1\rangle = \frac{1}{B\hbar}\langle -|\hat{\omega}|+\rangle|-\rangle \overset{*}{=} \frac{i\omega_0\cancel{\hbar}}{2}\frac{1}{B\cancel{\hbar}}|-\rangle = \frac{i}{2}|-\rangle\frac{\omega_0}{B}$$

Orduan: $\quad |\psi_2(\lambda)\rangle = |+\rangle + \lambda\frac{i}{2}\frac{\omega_0}{B}|-\rangle \Rightarrow |\psi_2(\lambda)\rangle = \frac{2B}{\sqrt{4B^2+\lambda^2\omega_0^2}}\left(|+\rangle + \frac{\lambda i\omega_0}{2B}|-\rangle\right)$

* Matrizeak

$$\varepsilon_1 = \langle +|\hat{\omega}|+\rangle \overset{*}{=} 0 \quad , \quad \varepsilon_2 = \sum_{m\neq n}{}'\sum_{1}\frac{|\langle\varphi_m^1|\hat{\omega}|\varphi_n\rangle|^2}{E_n^0 - E_m^0} = \frac{1}{B\hbar}|\langle -|\hat{\omega}|+\rangle|^2 \overset{*}{=}$$

$$\frac{1}{B\cancel{\hbar}}\frac{\omega_0^2\hbar^2}{4} = \frac{\omega_0^2\hbar}{4B} \Rightarrow E_2(\lambda) = E_1^0 + \frac{\omega_0^2\hbar}{4B}\lambda^2 = \frac{\hbar}{2}\left(\frac{3\hbar A}{2}+B\right) + \frac{\omega_0^2\hbar}{4B}\lambda^2$$

d) H hamiltondarraren ebazpen zehatza:

$$H_0 = AS^2 + BS_z \to (H_0) = A\frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\frac{3}{2}A\hbar+B & 0\\ 0 & \frac{3}{2}\hbar A - B\end{pmatrix}$$

$$(H) = \frac{\hbar}{2}\begin{pmatrix}\frac{3}{2}A\hbar+B & 0\\ 0 & \frac{3}{2}\hbar A-B\end{pmatrix} + i\frac{\hbar\omega_0}{2}\lambda\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\frac{3}{2}A\hbar+B & -i\omega_0\lambda\\ i\omega_0\lambda & \frac{3}{2}\hbar A-B\end{pmatrix}$$

($\{S^2, S_z\}$-ren aldibereko autobektoreen oinarrian; $\{|+\rangle, |-\rangle\}$)

$$|H - E\mathbb{1}| = \begin{vmatrix}\frac{\frac{3}{2}A\hbar^2+B\hbar}{2} - E & -i\omega_0\hbar/2\,\lambda\\ i\omega_0\hbar/2\,\lambda & \frac{\frac{3}{2}\hbar^2A - B\hbar}{2} - E\end{vmatrix} = \left(\frac{3A\hbar^2}{4} + \frac{B\hbar}{2} - E\right)\left(\frac{3\hbar^2 A}{4} - \frac{B\hbar}{2} - E\right) - \omega_0^2\frac{\hbar^2}{4}\lambda^2 = 0$$

$$\Rightarrow \frac{9\hbar^4A^2}{4^2} - \cancel{\frac{3AB\hbar^3}{8}} - \frac{3A\hbar^2E}{4} + \cancel{\frac{3B\hbar^3A}{8}} - \cancel{\frac{BE\hbar}{2}} - \frac{3EA\hbar^2}{4} + \cancel{\frac{B\hbar E}{2}} + E^2 - \omega_0^2\frac{\hbar^2\lambda^2}{4} - \frac{B^2\hbar^2}{4} = 0$$

$$E^2 - \frac{3}{2}A\hbar^2 E + \frac{\hbar^2}{4}\left(9\hbar^2 A^2 - B^2 - \omega_0^2\lambda^2\right) = 0 \rightarrow E = \frac{\frac{3}{2}A\hbar^2 \pm \sqrt{\cancel{\frac{9}{4}A^2\hbar^4} + \hbar^2\omega_0^2\lambda^2 - \cancel{9\hbar^4 A^2} + B^2\hbar^2}}{2} =$$

$$\frac{\frac{3}{2}A\hbar^2 \pm \hbar\sqrt{\omega_0^2\lambda^2 + B^2}}{2} \rightarrow E_1 = \frac{3A\hbar^2}{4} - \frac{\hbar}{2}\sqrt{\omega_0^2\lambda^2 + B^2}, \quad E_2 = \frac{3A\hbar^2}{4} + \frac{\hbar}{2}\sqrt{\omega_0^2\lambda^2 + B^2}$$

$*$ $E_1 \Rightarrow$

$$\begin{pmatrix} \frac{B\hbar}{2} + \frac{\hbar}{2}\sqrt{\omega_0^2\lambda^2 + B^2} & -i\omega_0\lambda\frac{\hbar}{2} \\ i\omega_0\lambda\frac{\hbar}{2} & -\frac{B\hbar}{2} + \frac{\hbar}{2}\sqrt{\omega_0^2\lambda^2 + B^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} B + \sqrt{\omega_0^2\lambda^2 + B^2} & -i\omega_0\lambda \\ i\omega_0\lambda & -B + \sqrt{\omega_0^2\lambda^2 + B^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$\frac{\hbar}{2}\begin{pmatrix} (B + \sqrt{\omega_0^2\lambda^2 + B^2})x - i\omega_0\lambda y \\ i\omega_0\lambda x - (B - \sqrt{\omega_0^2\lambda^2 + B^2})y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad i\omega_0\lambda y = (B + \sqrt{\omega_0^2 + \lambda^2 B^2})x \rightarrow$$

$$y = -i\frac{(B + \sqrt{\omega_0^2\lambda^2 + B^2})}{\omega_0\lambda}x \overset{*'}{\simeq} -i\frac{2B}{\omega_0\lambda}x$$

$*'$ $$\sqrt{\omega_0^2\lambda^2 + B^2} = B\sqrt{1 + \left(\frac{\omega_0\lambda}{B}\right)^2} \simeq B\left(1 + \frac{\omega_0^2\lambda^2}{2B^2}\right) \simeq B$$

$$\frac{\omega_0\lambda}{B} << 1$$

Orduan $\Rightarrow$ $$|\psi_1\rangle = |+\rangle - i\frac{2B}{\omega_0\lambda}|-\rangle \xrightarrow{\text{Normalizatu}} |\psi_1\rangle = \frac{2B}{\sqrt{\lambda^2\omega_0^2 + 4B^2}}\left(|-\rangle + i\frac{\omega_0\lambda}{2B}|+\rangle\right)$$

$*$ $E_2 \Rightarrow$

$$\begin{pmatrix} \frac{B\hbar}{2} - \frac{\hbar}{2}\sqrt{\omega_0^2\lambda^2 + B^2} & -i\omega_0\hbar\frac{\lambda}{2} \\ i\omega_0\lambda\frac{\hbar}{2} & -\frac{B\hbar}{2} - \frac{\hbar}{2}\sqrt{\omega_0^2\lambda^2 + B^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} B - \sqrt{\omega_0^2\lambda^2 + B^2} & -i\omega_0\lambda \\ i\omega_0\lambda & -B - \sqrt{\omega_0^2\lambda^2 + B^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$\frac{\hbar}{2}\begin{pmatrix} (B - \sqrt{\omega_0^2\lambda^2 + B^2})x - i\omega_0\lambda y \\ i\omega_0\lambda x - (B + \sqrt{\omega_0^2\lambda^2 + B^2})y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad i\omega_0\lambda x = (B + \sqrt{\omega_0^2\lambda^2 + B^2})\, y \Rightarrow$$

$$x = -\frac{i}{\omega_0\lambda}(B + \sqrt{\omega_0^2\lambda^2 + B^2}) \overset{*'}{\simeq} -i\frac{2B}{\omega_0\lambda}$$

Orduan $\Rightarrow$ $$|\psi_2\rangle = -i\frac{2B}{\omega_0\lambda}|+\rangle + |-\rangle \xRightarrow{\text{Normalizatu}} |\psi_2\rangle = \frac{2B}{\sqrt{4B^2 + \omega_0^2\lambda^2}}\left(|+\rangle + \frac{\lambda i\omega_0}{2B}|-\rangle\right)$$

Bai, emaitza berdinak lortzen dira.

5) Izan bedi $H = H_0 + \lambda \hat{W}$ hamiltondarra, non $H_0 = J_z^2 + \hbar J_z$ eta $\hat{W} = \hbar J_x$ diren. Lor bitez $\mathcal{E}(j=1)$ azpiespazioaren baitan, autobalioak (2. ordenako eta ordena altuagoko gaiak arbuiatuz) eta autobektoreak (lehen ordenako eta ordena altuagoko gaiak arbuiatuz)

$\mathcal{E}(j=1) \Rightarrow g=3 \rightarrow m=-1,0,1 \Rightarrow \{|1,-1\rangle, |1,0\rangle, |1,1\rangle\}$ oinarria.

Oinarri honetan: $(J_z) = \hbar \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(J_x) = \hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$,

$(J_y) = \hbar i \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

* $(H_0) = \hbar^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \hbar^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hbar^2 \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] =$

$\hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow$ Diagonala $\rightarrow$ Bere autobalioak: $E_1^0 = 0$ $(g=2)$, $E_2^0 = 2\hbar^2$

- $E_1^0 \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow c = 0$ $\begin{cases} |\varphi_1^{01}\rangle = |1,-1\rangle \\ |\varphi_1^{02}\rangle = |1,0\rangle \end{cases}$

- $E_2^0 \rightarrow \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a \\ -2b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow a = b = 0 \rightarrow |\varphi_2^0\rangle = |1,1\rangle$

* $(\hat{W}) = \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

H-k 3 autobektore eta 3 autobalio izango ditu:

- $E_1(\lambda) = \varepsilon_0 + \varepsilon_1 \lambda + \varepsilon_2 \lambda^2 + \dots$, $\varepsilon_0 = E_2^0 = 2\hbar^2$
- $|\Psi_1(\lambda)\rangle = |0\rangle + \lambda |1\rangle \rightarrow |0\rangle = |\varphi_2^0\rangle = |1,1\rangle$
  - $\varepsilon_1 = \langle \varphi_2^0 | \hat{W} | \varphi_2^0 \rangle = 0$ (matrizean)

$\circ \; |1\rangle = \sum_{m\neq n}{}' \sum_i \frac{\langle \varphi_m^i | \tilde{W} | \varphi_n \rangle}{E_n^0 - E_m^0} |\varphi_m^i\rangle = \frac{\langle \varphi_1^{01} | \tilde{W} | \varphi_2^0 \rangle}{2\hbar^2} |\varphi_1^{01}\rangle + \frac{\langle \varphi_1^{02} | \tilde{W} | \varphi_2^0 \rangle}{2\hbar^2} |\varphi_1^{02}\rangle =$

$\frac{1}{2\hbar^2}\left[ 0 + \frac{\hbar^2}{\sqrt{2}} |\varphi_1^{02}\rangle \right] = \frac{\cancel{\hbar^2}}{2\cancel{\hbar^2}\sqrt{2}} |1,0\rangle = \frac{1}{2\sqrt{2}} |1,0\rangle$

$\circ \; \varepsilon_2 = \sum_{m\neq n}{}' \sum_i \frac{|\langle \varphi_m^i | \tilde{W} | \varphi_n \rangle|^2}{E_n^0 - E_m^0} = \frac{|\langle \varphi_1^{01} | \tilde{W} | \varphi_2^0 \rangle|^2}{2\hbar^2} + \frac{|\langle \varphi_1^{02} | \tilde{W} | \varphi_2^0 \rangle|^2}{2\hbar^2} = \frac{\hbar^4}{2\hbar^2 \cdot 2} = \frac{\hbar^2}{4}$

Beraz $\Rightarrow \begin{cases} |\Psi_1(\lambda)\rangle = |1,1\rangle + \frac{\lambda}{2\sqrt{2}} |1,0\rangle \\ E(\lambda) = 2\hbar^2 + \frac{\lambda^2 \hbar^2}{4} = \hbar^2 \left(2 + \frac{\lambda^2}{4}\right) \end{cases}$

$\circ \; E_2(\lambda) = \varepsilon_0 + \varepsilon_1 \lambda + \varepsilon_2 \lambda^2 + \dots, \qquad \varepsilon_0 = E_1^0 = 0 \rightarrow |0\rangle = a|\varphi_1^1\rangle + b|\varphi_1^2\rangle$

$\tilde{W}^{(1)} \Rightarrow$ murrizketa : $(\tilde{W}^{(1)}) = \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$ Diagonalizatu: ENDAKAPENA

$|\tilde{W}^{(1)} - \varepsilon \mathbb{1}| = \begin{vmatrix} -\varepsilon & \frac{\hbar^2}{\sqrt{2}} \\ \frac{\hbar^2}{\sqrt{2}} & -\varepsilon \end{vmatrix} = +\varepsilon^2 - \frac{\hbar^4}{2} = 0 \rightarrow \varepsilon = \pm \frac{\hbar^2}{\sqrt{2}}$

$* \; \varepsilon_1^1 = \frac{\hbar^2}{\sqrt{2}} \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a+b \\ a-b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow |0\rangle_1 = \frac{1}{\sqrt{2}} [|1,-1\rangle + |1,0\rangle]$

$|1\rangle_1 = \sum_{m\neq n}{}' \sum_i \frac{\langle \varphi_m^i | \tilde{W} | 0 \rangle}{E_n^0 - E_m^0} |\varphi_m^i\rangle = \frac{\langle \varphi_2^0 | \tilde{W} | 0 \rangle}{-2\hbar^2} |\varphi_2^0\rangle = \frac{-1}{2\hbar^2} \left[ \langle \varphi_2^0 | \tilde{W} | \varphi_1^{01} \rangle \cdot \frac{1}{\sqrt{2}} + \right.$

$\left. \frac{1}{\sqrt{2}} \langle \varphi_2^0 | \tilde{W} | \varphi_1^{02} \rangle \right] |\varphi_2^0\rangle = \frac{-\hbar^2}{2\sqrt{2}} \left[ 0 + \hbar^2 \right] |\varphi_2^0\rangle = \frac{-1}{2\sqrt{2}} |\varphi_2^0\rangle$

$\circ \; \varepsilon_2 = \sum_{m\neq n}{}' \sum_i \frac{|\langle \varphi_m^i | \tilde{W} | 0 \rangle|^2}{E_n^0 - E_m^0} = -\frac{1}{2\hbar^2} \left[ |\langle \varphi_2^0 | \tilde{W} | 0 \rangle|^2 \right] = -\frac{1}{2\hbar^2} \left[ |\langle \varphi_2^0 | \tilde{W} | \varphi_1^{01} \rangle \cdot \frac{1}{\sqrt{2}} + \right.$

$\left. \frac{1}{\sqrt{2}} \langle \varphi_2^0 | \tilde{W} | \varphi_1^{02} \rangle |^2 \right] = -\frac{1}{2\hbar^2} \left[ \frac{\hbar^2}{\sqrt{2}} \right]^2 = \frac{-\hbar^4}{4\hbar^2} = -\frac{\hbar^2}{4}$

Orduan: $E_2(\lambda) = \hbar^2 \lambda - \frac{\hbar^2}{4} \lambda^2, \qquad |\Psi_2(\lambda)\rangle = \frac{1}{\sqrt{2}} [|1,-1\rangle + |1,0\rangle] - \frac{\lambda}{2\sqrt{2}} |1,1\rangle$

$* \ \varepsilon_1{}^2 = -\hbar^2 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to a=-b \Rightarrow |0\rangle_1 = \frac{1}{\sqrt{2}}\left[|1,-1\rangle - |1,0\rangle\right]$

$$|1\rangle_1 = \sum_{m\neq n}{}' \sum_i \frac{\langle \varphi_m^i|\tilde{\omega}|\varphi_n\rangle}{E_n^0 - E_m^0}|\varphi_m^i\rangle = -\frac{1}{2\hbar^2}\langle\varphi_2^0|\tilde{\omega}|0\rangle_2 |\varphi_2^0\rangle = -\frac{1}{2\hbar^2}|\varphi_2^0\rangle \cdot \frac{1}{\sqrt{2}}$$

$$\left[\langle\varphi_2^0|\tilde{\omega}|\varphi_1^1\rangle - \langle\varphi_2^0|\tilde{\omega}|\varphi_1^2\rangle\right] = -\frac{1}{2\sqrt{2}\hbar^2}|\varphi_2^0\rangle\left[-\hbar^2\right] = \frac{1}{2\sqrt{2}}|\varphi_2^0\rangle$$

$$\circ \ \varepsilon_2 = \sum_{m\neq n}{}' \sum_i \frac{|\langle\varphi_m^i|\tilde{\omega}|\varphi_n\rangle|^2}{E_n^0 - E_m^0} = -\frac{1}{2\hbar^2}|\langle\varphi_2^0|\tilde{\omega}|0\rangle_2|^2 = -\frac{1}{2\hbar^2}\Big[|\langle\varphi_2^0|\tilde{\omega}|\varphi_1^1\rangle\frac{1}{\sqrt{2}} +$$

$$-\langle\varphi_2^0|\tilde{\omega}|\varphi_1^2\rangle\frac{1}{\sqrt{2}}|^2\Big] = -\frac{1}{2\hbar^2}\left[-\frac{\hbar^2}{\sqrt{2}}\right]^2 = -\frac{\hbar^4}{2\hbar^2} = -\frac{\hbar^2}{2}$$

Orduan $\Rightarrow E_3(\lambda) = -\hbar^2\lambda - \frac{\hbar^2}{2}\lambda^2$, $|\psi_3(\lambda)\rangle = \frac{1}{\sqrt{2}}\left[|1,-1\rangle - |1,0\rangle\right] + \frac{\lambda}{2\sqrt{2}}|1,1\rangle$

6.) Izan bedi $H = H_0 + \lambda\tilde{\omega}$ hamiltondarra, non $H_0 = J_z^2$ eta $\tilde{\omega} = J_x^2$. Lor bitez, $\varepsilon(j=1)$ azpiespazioaren baitan, autobalioak (bigarren ordenako eta ordena altuagoko gaiak arbuiatuz) eta autobektoreak (lehen ordenako eta ordena altuagoko gaiak arbuiatuz).

$\varepsilon(j=1) \Rightarrow g=3 \to m=-1,0,1 \Rightarrow \{|1,-1\rangle, |1,0\rangle, |1,1\rangle\}$ oinarria

Oinarri horretan: $(J_z) = \hbar\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ; $(J_x) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$* \ (H_0) = \hbar^2\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to$ Diagonala: $E_1^0 = 0$, $E_2^0 = \hbar^2 \ (g=2)$

(autobalioak)

$|\varphi_1^0\rangle = |1,0\rangle$, $|\varphi_2^{0\,1}\rangle = |1,-1\rangle$, $|\varphi_2^{0\,2}\rangle = |1,1\rangle$

(autobektoreak)

$* \quad |\tilde{W}\rangle = \hbar^2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad *'$

H-K 3 autobektore izango ditu:

- $E_1(\lambda)$ , $|\Psi_1(\lambda)\rangle$ : $E_1(\lambda) = \mathcal{E}_0 + \mathcal{E}_1 \lambda + \dots$ ; $|\Psi_1(\lambda)\rangle = |0\rangle + \dots$ $*'$

$\mathcal{E}_0 = E_1^0 = 0$ , $|0\rangle = |\varphi_1^0\rangle = |1,0\rangle$ , $\mathcal{E}_1 = \langle 0|\hat{W}|0\rangle = \langle 1,0|\tilde{W}|1,0\rangle =$

$\hbar^2 \Rightarrow E_1(\lambda) = \lambda \hbar^2$ , $|\Psi_1(\lambda)\rangle = |1,0\rangle$

- $E_2(\lambda)$ , $|\Psi_2(\lambda)\rangle$ : $E_2(\lambda) = \mathcal{E}_0 + \mathcal{E}_1 \lambda + \dots$ , $|\Psi_1(\lambda)\rangle = |0\rangle$

$\mathcal{E}_0 = E_2^0 = \hbar^2$ (endekatua $\Rightarrow$ $\hat{W}^{(2)}$ diagonalizatu) $\hat{W}^{(2)} = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\hat{W}^{(2)}$ diagonalizatu $\Rightarrow$ $\begin{vmatrix} \frac{\hbar^2}{2} - \mathcal{E}_j & \frac{\hbar^2}{2} \\ \frac{\hbar^2}{2} & \frac{\hbar^2}{2} - \mathcal{E}_j \end{vmatrix} = \left(\frac{\hbar^2}{2} - \mathcal{E}_j\right)^2 - \frac{\hbar^4}{4} = 0 \Rightarrow \frac{\hbar^2}{2} - \mathcal{E}_j = \pm \frac{\hbar^2}{2} \Rightarrow$

$\mathcal{E}_1^1 = 0$ , $\mathcal{E}_1^2 = \hbar^2$

$* \mathcal{E}_1^1 = 0 \Rightarrow \begin{pmatrix} \hbar^2 & \hbar^2 \\ \hbar^2 & \hbar^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \hbar^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \hbar^2 \begin{pmatrix} a+b \\ a+b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b$

$|0\rangle_1 = \frac{1}{\sqrt{2}} [|\varphi_2^{0\,1}\rangle - |\varphi_2^{0\,2}\rangle] = \frac{1}{\sqrt{2}} [|1,-1\rangle - |1,1\rangle]$

$* \mathcal{E}_1^2 = 2\hbar^2 \Rightarrow \begin{pmatrix} -\hbar^2 & \hbar^2 \\ \hbar^2 & -\hbar^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \hbar^2 \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \hbar^2 \begin{pmatrix} -a+b \\ a-b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b$

$|0\rangle_2 = \frac{1}{\sqrt{2}} [|\varphi_2^{0\,1}\rangle + |\varphi_2^{0\,2}\rangle] = \frac{1}{\sqrt{2}} [|1,-1\rangle + |1,1\rangle]$

$E_2(\lambda) = E_2^0 + \mathcal{E}_1^1 = \lambda\hbar^2$ , $|\Psi_2(\lambda)\rangle = |0\rangle_1 = \frac{1}{\sqrt{2}} [|1,-1\rangle - |1,1\rangle]$

- $E_3(\lambda)$ , $|\Psi_3(\lambda)\rangle \Rightarrow$ $E_3(\lambda) = E_2^0 + \lambda\mathcal{E}_1^2 = \hbar^2 + 2\hbar^2\lambda = \hbar^2(1+2\lambda)$

$|\Psi_3(\lambda)\rangle = |0\rangle_2 = \frac{1}{\sqrt{2}} [|1,-1\rangle + |1,1\rangle]$

7.) Izan bedi OXY planoan R erradudo zirkunferentzia bati jarraituz higitzen ari den m masako eta q kargako partikula.

Planoan $\hat{L}_z^2 = \hat{L}^2$

a) $H_0 = \dfrac{\hat{L}_z^2}{2mR^2}$ $\Rightarrow$ $H_0$-ren autobektoreak $\hat{L}_z^2$-renak eta autobalioak $\dfrac{\hbar^2 m_l^2}{2mR^2}$

$E^0_{m_l} = \dfrac{\hbar^2 m_l^2}{2mR^2}$ , $|\varphi^0_{m_l}\rangle = |m_l\rangle$ (Askatasun gradu bakarra du, $\varphi$)

Endakapena $\to$ $\pm|m_l|$-k energia bera dute $\to g = 2(2l+1)$

b) $H = H_0 + W$ , $W = -qE_0R\cos\varphi$ $\to$ Perturbazioen teoria. $m_l = 0$ ez dago endakatuta.

$E(\lambda) = \varepsilon_0 + \lambda\varepsilon_1 + \lambda^2\varepsilon_2 + O(\lambda^3)$ , $|\psi(\lambda)\rangle = |0\rangle + \lambda|1\rangle + O(\lambda^2)$

$|m_l|$ zehaztuta dagoenean $\to$ bi aukera $\to \pm m_l$

$\varepsilon_0 = E^0_{|m_l|}$ (endakatua) $\Rightarrow$ $W^{(m_l)}$ diagonalizatu $(|m_l| \neq 1)$

$*$ $\langle m_l | W | m_l\rangle = -qE_0R\langle m_l|\cos\varphi|m_l\rangle = -\dfrac{qE_0R}{2\pi}\int_0^{2\pi} \cancel{e^{-im_l\varphi}}\cos\varphi \cancel{e^{im_l\varphi}}\,d\varphi =$

$-\dfrac{qE_0R}{2\pi}\int_0^{2\pi}\cos\varphi\,d\varphi = -\dfrac{qE_0R}{2\pi}\left[\,[\sin\varphi]_0^{2\pi}\right] = 0$

$*$ $\langle |m_l| \,|\, W \,|\, -|m_l|\rangle = -\dfrac{qE_0R}{2\pi}\int_0^{2\pi} e^{-i|m_l|\varphi}\cos\varphi\, e^{-i|m_l|\varphi}\,d\varphi =$

$-\dfrac{qE_0R}{2\pi}\int_0^{2\pi} e^{-2i|m_l|\varphi}\left(\dfrac{e^{i\varphi}+e^{-i\varphi}}{2}\right)d\varphi = -\dfrac{qE_0R}{4\pi}\int_0^{2\pi} e^{i\varphi(1-2|m_l|)} + e^{-i\varphi(1+2|m_l|)}\,d\varphi =$

$-\dfrac{qE_0R}{4\pi}\left[\left[\dfrac{1}{i(1-2|m_l|)}e^{i\varphi(1-2|m_l|)}\right]_0^{2\pi} + \left[\dfrac{e^{-i\varphi(1+2|m_l|)}}{-i(1+2|m_l|)}\right]_0^{2\pi}\right] =$

$+\dfrac{qE_0Ri}{4\pi}\left[\dfrac{e^{2\pi i(1-2|m_l|)}-1}{1-2|m_l|} + \dfrac{1-e^{-2\pi i(1+2|m_l|)}}{1+2|m_l|}\right] = 0$ $(|m_l|\in\mathbb{N})$

$*$ $\langle -|m_l| \,|\, W \,|\, |m_l|\rangle = -\dfrac{qE_0R}{2\pi}\int_0^{2\pi} e^{i|m_l|\varphi}\cos\varphi\, e^{i|m_l|\varphi}\,d\varphi = -\dfrac{qE_0R}{2\pi}\int_0^{\pi}\cos\varphi\, e^{2i|m_l|\varphi}\,d\varphi =$

$-\dfrac{qE_0R}{4\pi}\int_0^{2\pi}\left(e^{i\varphi(1+2|m_l|)} + e^{-i\varphi(1-2|m_l|)}\right)d\varphi = i\dfrac{qE_0R}{4\pi}\left[\dfrac{e^{2\pi i(1+2|m_l|)}-1}{1+2|m_l|} + \dfrac{1-e^{-2\pi i(1-2|m_l|)}}{1-2|m_l|}\right] = 0$

$*^2$ 1c)

$$H = H_0 + \lambda \frac{1}{2} m\omega^2 x^2 = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \frac{\lambda}{2} m\omega^2 x^2 = \frac{p_x^2}{2m} + \frac{1}{2} m \big(\underbrace{\omega\sqrt{1+\lambda}}_{\omega'}\big)^2 x^2$$

$$\omega' = \omega\sqrt{1+\lambda} \Rightarrow H|\psi_n\rangle = \varepsilon|\psi_n\rangle$$

Taylor

$$\varepsilon = \hbar\omega'(n+1/2) = \hbar\omega\sqrt{1+\lambda}\,(n+1/2) \simeq \hbar\omega(n+1/2)\left(1+\frac{\lambda}{2}-\frac{\lambda^2}{8}+O(\lambda^3)\right)$$

Aurreliaoren bardina, perturbazioa teoriaren korrekzioa

$$\langle x|\psi_0\rangle_\omega = \langle x|0\rangle_{\omega(\sqrt{1+\lambda})} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}(1+\lambda)^{1/8} e^{-\sqrt{1+\lambda}\,\tilde{x}^2/2} \simeq$$

→ Taylor

$$\langle x|\psi_0\rangle = \langle x|0\rangle - \frac{\lambda}{8}\sqrt{2}\langle x|2\rangle$$

$$\hookrightarrow \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\tilde{x}^2/2}$$

$$\text{Taylor} \Rightarrow (1+\lambda)^{1/8} \simeq \left(1+\frac{\lambda}{8}+\dots\right) \quad / \quad e^{-\sqrt{1+\lambda}\,\tilde{x}^2/2} \simeq e^{-\tilde{x}^2/2} \cdot e^{-\lambda \tilde{x}^2/4} \simeq$$

$$e^{-\tilde{x}^2/2}\left(1-\lambda\frac{\tilde{x}^2}{4}\right)$$

$$W^{|m_l|} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow |W^{|m_l|} - \varepsilon_1 \mathbb{1}| = \begin{vmatrix} -\varepsilon_1 & 0 \\ 0 & -\varepsilon_1 \end{vmatrix} = \varepsilon_1^2 = 0 \rightarrow \varepsilon_1 = 0$$

(bikoitza)

$\varepsilon_1 = 0 \rightarrow \quad |0\rangle_1 = |m_l\rangle \quad , \quad |0\rangle_2 = |-m_l\rangle$ Ezin da zehaztu

$$\varepsilon_2^{\pm} = \sum_{m' \neq m}{}' \sum_{j} \frac{|\langle m_l'^{j} | \tilde{W} | \pm m_l \rangle|^2}{E_{m'l} - E_{ml}} \overset{*}{=} 0$$

+ edo −

?

+ edo −

$$* \quad \langle m_l'^{j} | \tilde{W} | m'_l \rangle = -\frac{q E_0 R}{2\pi} \int_0^{2\pi} e^{-i m_l'^{j}} e^{i m'_l} \cos\varphi \, d\varphi = \frac{q E_0 R}{2\pi} \cdot 0$$

8.)

$$H = \underbrace{\frac{\hbar^2}{2m} \nabla^2 - \frac{Ze^2}{4\pi\varepsilon_0 r^2}}_{H_0} - \underbrace{\frac{e^2}{4\pi\varepsilon_0 r^2}}_{W} \Rightarrow$$

Aplikatu perturbazioen teoria

Soluzio zehatza (Z+1) jarriz.

# 4. ERRADIAZIO ELEKTROMAGNETIKOA

16-11-19

4.1)

$$\vec{A}' = \vec{A} + \vec{\nabla}\psi \quad , \quad \phi' = \phi - \frac{\partial\psi}{\partial t}$$ (Kontrastello transformazioak)

a) $$\vec{E}' = -\vec{\nabla}\phi' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\phi + \vec{\nabla}\left(\frac{\partial\psi}{\partial t}\right) - \frac{\partial\vec{A}}{\partial t} - \frac{\partial}{\partial t}(\vec{\nabla}\psi) = -\vec{\nabla}\phi + \cancel{\frac{\partial}{\partial t}(\vec{\nabla}\psi)} - \frac{\partial\vec{A}}{\partial t} - \cancel{\frac{\partial}{\partial t}(\vec{\nabla}\psi)} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} = \vec{E}$$

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A}+\vec{\nabla}\psi) = \vec{\nabla}\times\vec{A} + \cancel{\vec{\nabla}\times(\vec{\nabla}\psi)} = \vec{\nabla}\times\vec{A} = \vec{B}$$

b) Lorentzen baldintza: $\vec{\nabla}\cdot\vec{A}' + \varepsilon\mu\frac{\partial\phi'}{\partial t} = 0$ $\quad$ ( $\vec{\nabla}\cdot\vec{A} + \varepsilon\mu\frac{\partial\phi}{\partial t} = 0$ betetzen dute)

$$\vec{\nabla}\cdot(\vec{A}+\vec{\nabla}\psi) + \varepsilon\mu\frac{\partial}{\partial t}\left(\phi - \frac{\partial\psi}{\partial t}\right) = \cancel{\vec{\nabla}\cdot\vec{A}} + \nabla^2\psi + \cancel{\varepsilon\mu\frac{\partial\phi}{\partial t}} - \varepsilon\mu\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - \varepsilon\mu\frac{\partial^2\psi}{\partial t^2} = 0 \Rightarrow$$

$$\nabla^2\psi = \varepsilon\mu\frac{\partial^2\psi}{\partial t^2}$$

4.2)

Hutsean, $\varepsilon = \varepsilon_0$; $\mu = \mu_0$ eta $\vec{J} = 0$; Coulomb-en kontrastea, "gauge"?: $\begin{cases} \vec{\nabla}\cdot\vec{A} = 0 \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \end{cases}$

- $\vec{\nabla}\cdot\vec{D} = \rho$ , $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = -\nabla^2\phi - \cancel{\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial t}} = \frac{\rho}{\varepsilon_0} = -\nabla^2\phi = \frac{\rho}{\varepsilon_0} \Rightarrow$

$$\nabla^2\phi = -\rho/\varepsilon_0$$

- $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$
- $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right) = -\cancel{\vec{\nabla}\times\vec{\nabla}\phi} - \vec{\nabla}\times\frac{\partial\vec{A}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$
- $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \Leftrightarrow \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{1}{\mu_0}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{1}{\mu_0}\left(\vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{A}}^{0}) - \nabla^2\vec{A}\right) = -\frac{\nabla^2\vec{A}}{\mu_0} =$

$$\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right) = \varepsilon_0\left[-\vec{\nabla}\frac{\partial\phi}{\partial t} - \frac{\partial^2\vec{A}}{\partial t^2}\right] \Rightarrow 0 = \nabla^2\vec{A} - \frac{1}{c^2}\left(\vec{\nabla}\frac{\partial\phi}{\partial t} + \frac{\partial^2\vec{A}}{\partial t^2}\right) = -\vec{\nabla}\frac{\partial\phi}{\partial t}\cdot\frac{1}{c^2} \Rightarrow$$

$$\vec{\nabla}\cdot\left(\frac{\partial\phi}{\partial t}\right) = 0$$

4.3.1

$\Psi(r)$ simetria esferikoko funtzio eskalarra $\Rightarrow$ uhin-ekuazioa?

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \iff \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} =$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t} = 0 \Rightarrow \frac{1}{r} f(r\pm ct) = \psi(r)$$

Froga: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{1}{r^2}f(r\pm ct) + \frac{1}{r}\frac{\partial f}{\partial r}\right)\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t}\left(\frac{1}{r}f(r\pm ct)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-f(r\pm ct) + r\frac{\partial f}{\partial r}\right) +$

$$-\frac{1}{rc^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{r^2}\left(-\cancel{\frac{\partial f}{\partial r}} + \cancel{\frac{\partial f}{\partial r}} + r\frac{\partial^2 f}{\partial r^2}\right) - \frac{1}{rc^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{r}\frac{\partial^2 f}{\partial r^2} - \frac{1}{rc^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{r}\left[\frac{\partial^2 f}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}\right] =$$

$$\frac{1}{r}\left[f'' - \frac{\cancel{c^2}}{\cancel{c^2}}f''\right] = 0$$

$$* \quad \frac{\partial(f(r\pm ct))}{\partial t} = \frac{\partial f(r\pm ct)}{\partial(r\pm ct)}\cdot\frac{\partial(r\pm ct)}{\partial t} = f'(-c)$$

4.4.1

Oinarrizko dipolo aldakorra:

a) Eskualde hurbila, quasiestatikoa $\Rightarrow \vec{B} = 0 \iff \langle u_{mag}\rangle = 0 \Rightarrow \frac{\langle u_{mag}\rangle}{\langle u_{elek}\rangle} = 0$

b) Tarteko, indukziozko eskualdea $\Rightarrow E_r = \frac{2\cos\theta[\dot{p}]}{4\pi\varepsilon_0 r^2 c}$ , $E_\theta = \frac{\sin\theta[\dot{p}]}{4\pi\varepsilon_0 r^2 c} \Rightarrow$

$$\frac{\langle E^2\rangle\varepsilon_0}{2} = \langle u_{elek}\rangle = \frac{\varepsilon_0}{2}\langle[E_r^2 + E_\theta^2]\rangle = \frac{\varepsilon_0}{2}\left[\frac{4\cos^2\theta\langle[\dot{p}]^2\rangle}{(4\pi\varepsilon_0 c)^2 r^4} + \frac{\sin^2\theta\langle[\dot{p}]^2\rangle}{(4\pi\varepsilon_0 c)^2 r^4}\right] = \frac{\varepsilon_0}{2r^4}\frac{\langle[\dot{p}]^2\rangle}{(4\pi\varepsilon_0 c)^2}[4\cos^2\theta + \sin^2\theta]$$

$$B = B_\varphi = \frac{\mu_0}{4\pi}\sin\theta\frac{[\dot{p}]}{r^2} \Rightarrow \langle u_{mag}\rangle = \frac{1}{2\mu_0}\langle B^2\rangle = \frac{1}{2\mu_0}\frac{\mu_0^2}{(4\pi)^2}\frac{\sin^2\theta\langle[\dot{p}]^2\rangle}{r^4}$$

$$\frac{\langle u_{mag}\rangle}{\langle u_{elek}\rangle} = \frac{\frac{\mu_0}{2}\frac{\sin^2\theta}{(4\pi)^2 r^4}[\dot{p}]^2}{\frac{\varepsilon_0}{2r^4}\frac{[\dot{p}]^2}{(4\pi)^2\varepsilon_0^2}\frac{[4\cos^2\theta + \sin^2\theta]}{c^2}} = \mu_0\varepsilon_0 c^2\frac{\sin^2\theta}{(4\cos^2\theta + \sin^2\theta)} = \frac{\sin^2\theta}{4\cos^2\theta + \sin^2\theta} = f(\theta)$$

c) Urruneko, erradiaziozko eskualdea $\Rightarrow E_r = 0$ , $E_\theta = \frac{\sin\theta[\ddot{p}]}{4\pi\varepsilon_0 rc^2} \Rightarrow \langle u_{elek}\rangle = \frac{1}{2}\varepsilon_0\langle E^2\rangle = \frac{1}{2}\varepsilon_0\langle E_\theta^2\rangle =$

$$\frac{1}{2}\cancel{\varepsilon_0}\frac{\sin^2\theta\langle[\ddot{p}]^2\rangle}{(4\pi)^2\varepsilon_0^{\cancel{2}} r^2 c^4} = \frac{1}{2}\frac{\sin^2\theta\langle[\ddot{p}]^2\rangle}{(4\pi)^2\varepsilon_0 r^2 c^4} \quad ; \quad B = B_\varphi = \frac{\mu_0}{4\pi}\sin\theta\frac{[\ddot{p}]}{rc} \Rightarrow \langle u_{mag}\rangle = \frac{1}{2\mu_0}\langle B^2\rangle =$$

$$\frac{1}{2\cancel{\mu_0}}\frac{\mu_0^{\cancel{2}}}{(4\pi)^2}\frac{\sin^2\theta\langle[\ddot{p}]^2\rangle}{r^2c^2} \Rightarrow \frac{\langle u_{mag}\rangle}{\langle u_{elek}\rangle} = \frac{\cancel{\frac{1}{2}}\frac{\mu_0}{\cancel{(4\pi)^2}}\frac{\cancel{\sin^2\theta}\cancel{[\ddot{p}]^2}}{\cancel{r^2}c^2}}{\cancel{\frac{1}{2}}\frac{\cancel{\sin^2\theta}\cancel{[\ddot{p}]^2}}{\cancel{(4\pi)^2}\varepsilon_0\cancel{r^2}c^4}} = \varepsilon_0\mu_0 c^2 = 1$$

4.5.1

Oinarrizko dipoloaren erradiazioko eremuak: $E_r = 0$, $E_\theta = \frac{\sin\theta}{4\pi\varepsilon_0}\frac{[\ddot{p}]}{rc^2} = E$, $B_\varphi = \frac{\mu_0}{4\pi}\frac{\sin\theta\,[\ddot{p}]}{rc} = B$

Frogatu uhin ekuazioa betetzen dutela:

- $\vec{E} = \frac{\sin\theta}{4\pi\varepsilon_0}\frac{[\ddot{p}]}{rc^2}\hat{u}_\theta \Rightarrow \nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t^2} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\vec{E}}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\vec{E}}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} =$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta\,[\ddot{p}]}{4\pi\varepsilon_0 rc^2}\cos\theta\,\hat{u}_\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\left(-\frac{1}{c}\right)\frac{\sin\theta}{4\pi\varepsilon_0}\cdot\frac{[\dddot{p}]}{rc^2} - \frac{\sin\theta\,[\ddot{p}]}{4\pi\varepsilon_0 r^2c^2}\right]\hat{u}_\theta\right) - \frac{1}{c^2}\frac{\sin\theta\,[\ddddot{p}]\,\hat{u}_\theta}{4\pi\varepsilon_0 rc^2} =$$

$$\frac{1}{r^2\sin\theta}\frac{[\ddot{p}]}{4\pi\varepsilon_0 rc^2}(\cos^2\theta - \sin^2\theta)\,\hat{u}_\theta - \frac{1}{r^2}\frac{\partial}{\partial r}\left(\left(\frac{r\sin\theta}{4\pi\varepsilon_0}\frac{[\dddot{p}]}{c^3} + \frac{\sin\theta\,[\ddot{p}]}{4\pi\varepsilon_0 c^2}\right)\hat{u}_\theta\right) - \frac{\sin\theta\,[\ddddot{p}]\,\hat{u}_\theta}{4\pi\varepsilon_0 rc^4} =$$

$$\frac{[\ddot{p}]}{r^3\sin\theta\,4\pi\varepsilon_0 c^2}(\cos^2\theta - \sin^2\theta)\,\hat{u}_\theta - \frac{1}{r^2}\left(\cancel{\frac{\sin\theta\,[\dddot{p}]}{4\pi\varepsilon_0 c^3}} - \frac{r\sin\theta\,[\ddddot{p}]}{4\pi\varepsilon_0 c^4} - \cancel{\frac{\sin\theta\,[\dddot{p}]}{4\pi\varepsilon_0 c^3}}\right)\hat{u}_\theta - \frac{\sin\theta\,[\ddddot{p}]}{4\pi\varepsilon_0 rc^4}\hat{u}_\theta =$$

$$\frac{[\ddot{p}]}{r^3\sin\theta\,4\pi\varepsilon_0 c^2}(\cos^2\theta - \sin^2\theta)\,\hat{u}_\theta + \cancel{\frac{\sin\theta\,[\ddddot{p}]}{r4\pi\varepsilon_0 c^4}}\hat{u}_\theta - \cancel{\frac{\sin\theta\,[\ddddot{p}]}{4\pi\varepsilon_0 rc^4}}\hat{u}_\theta = \frac{[\ddot{p}]}{r^2 4\pi\varepsilon_0 c^2}\left(\frac{\cos^2\theta}{\sin\theta} - \sin\theta\right)\hat{u}_\theta =$$

- $\vec{B} = \frac{\sin\theta\,\mu_0}{4\pi rc}[\ddot{p}]\,\hat{u}_\varphi \Rightarrow \nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial\vec{B}}{\partial t^2} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\vec{B}}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\vec{B}}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} =$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta\cos\theta\,\mu_0}{4\pi rc}[\ddot{p}]\,\hat{u}_\varphi\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[-\frac{1}{c}\frac{\sin\theta\,\mu_0[\dddot{p}]}{4\pi rc} - \frac{1}{r^2}\frac{\sin\theta\,\mu_0[\ddot{p}]}{4\pi c}\right]\hat{u}_\varphi\right) - \frac{1}{c^2}\frac{\sin\theta\,\mu_0}{4\pi rc}[\ddddot{p}]\,\hat{u}_\varphi =$$

$$\frac{1}{r^2\sin\theta}\cdot\frac{\mu_0[\ddot{p}]}{4\pi rc}(\cos^2\theta - \sin^2\theta)\,\hat{u}_\varphi - \frac{1}{r^2}\frac{\partial}{\partial r}\left(\left(\frac{\sin\theta\,\mu_0[\dddot{p}]\,r}{4\pi c^2} + \frac{\sin\theta\,\mu_0[\ddot{p}]}{4\pi c}\right)\hat{u}_\varphi\right) - \frac{1}{c^2}\frac{\sin\theta\,\mu_0}{4\pi rc}[\ddddot{p}]\,\hat{u}_\varphi =$$

$$\frac{\mu_0[\ddot{p}]}{4\pi r^3c\sin\theta}(\cos^2\theta - \sin^2\theta)\,\hat{u}_\varphi - \frac{1}{r^2}\left(-\frac{\sin\theta\,\mu_0[\ddddot{p}]\,r}{4\pi c^3} + \cancel{\frac{\sin\theta\,\mu_0[\dddot{p}]}{4\pi c^2}} - \cancel{\frac{\sin\theta\,\mu_0[\dddot{p}]}{4\pi c^2}}\right)\hat{u}_\varphi - \frac{\sin\theta\,\mu_0[\ddddot{p}]}{4\pi rc^3}\hat{u}_\varphi =$$

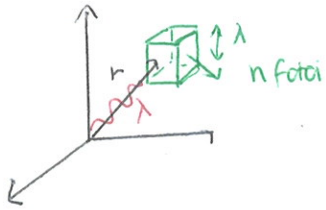
$$\frac{\mu_0[\ddot{p}]}{4\pi r^3c\sin\theta}(\cos^2\theta - \sin^2\theta)\,\hat{u}_\varphi + \cancel{\frac{\sin\theta\,\mu_0[\ddddot{p}]}{4\pi c^3r}}\hat{u}_\varphi - \cancel{\frac{\sin\theta\,\mu_0[\ddddot{p}]}{4\pi c^3r}}\hat{u}_\varphi = \frac{\mu_0[\ddot{p}]}{4\pi r^3c\sin\theta}(\cos^2\theta - \sin^2\theta)\,\hat{u}_\varphi =$$

$$\frac{\mu_0[\ddot{p}]}{4\pi r^3c}\left(\frac{\cos^2\theta}{\sin\theta} - \sin\theta\right)\hat{u}_\varphi$$

4.6.]

FM $\Rightarrow$ $\nu = 100$ MHz $\quad P = 10\,kW = 10^4\,W$ $\quad$ ( eradiazio isotropoa $\Rightarrow$ norabide guztietan era

berean hedatzen delal ; $\lambda = \frac{c}{\nu} = 3m$ , $r = 1\,km \gg \lambda$ Erradiazioko zatean

- 



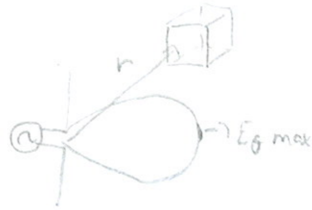
$$n = \frac{E_{totala}}{E_{fotoi}} = \frac{\langle U_{elektromag} \rangle}{h\nu} = \frac{\langle u_{em} \rangle V}{h\nu} = \frac{\langle S \rangle V}{c h\nu} =$$

(energia; isotropoa)

$$\frac{P \cdot V}{4\pi r^2 \cdot c h\nu} = \frac{P \cdot \lambda^3}{4\pi r^2 \cdot \lambda h \nu^2} = \frac{P\lambda^2}{4\pi r^2 h \nu^2} = 1'08 \cdot 10^{15} \text{ fotoi} \simeq 10^{15} \text{ fotoi}$$

$$\langle S \rangle = \frac{P}{4\pi r^2} = \frac{1}{2} c \varepsilon_0 E_0^2 \Rightarrow E_0 = \sqrt{\frac{P}{2\pi r^2 \cdot c \varepsilon_0}} = 0'77 \ V/m$$

- $\lambda/2 = L$





$$\langle P \rangle = 73'1 \frac{I_0^2}{2} \Rightarrow I_0 = \sqrt{\frac{2\langle P \rangle}{73'1}} = 16'54\ A$$

$$(E_\theta)_{max} = \frac{1}{4\pi\varepsilon_0} \frac{I_0}{rc} \cdot 2 = 0'99 \ V/m$$

( energia hau bidaltzen duso , haregatik dira maximo edo minimoak )

$$\langle S \rangle \Rightarrow \langle u_{EM} \rangle = \frac{\langle S \rangle}{c} = \frac{1}{c} \cdot \frac{1}{8\pi^2 \varepsilon_0 c} \frac{I_0^2}{r^2} = \frac{I_0^2}{8\pi^2 \varepsilon_0 c^2 r^2}$$

(maximoa)

$$n = \frac{\langle u_{em} \rangle V}{h\nu} = \frac{1}{h\nu} \cdot \frac{I_0^2}{8\pi^2 \varepsilon_0 c^2 r^2} \cdot \lambda^3 = \frac{I_0^2 \, c}{h\nu^4 \, 8\pi^2 \varepsilon_0 r^2} = 1'77 \cdot 10^{15} \text{ fotoi} \simeq 2 \cdot 10^{15} \text{ fotoi}$$

- $L \ll \lambda$ $\quad$ Dipolo elektrikoa (erradiazioarekoa)

$$n = \frac{\langle U_{elektromag} \rangle}{h\nu} = \frac{\langle u_{em} \rangle V}{h\nu} = \frac{\langle S \rangle V}{c h\nu} = \frac{\langle [\ddot{p}]^2 \rangle}{16\pi^2} \frac{V}{\varepsilon_0 c^3 r^2} \cdot \frac{1}{c h\nu} = \frac{6\pi\varepsilon_0 c^3 \langle P \rangle V}{16\pi^2 \varepsilon_0 c^4 r^2 h\nu} =$$

(maximoa)

$$\langle P \rangle = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2}{3} \frac{\langle [\ddot{p}]^2 \rangle}{c^3} \Rightarrow \langle [\ddot{p}]^2 \rangle = 6\pi\varepsilon_0 c^3 \langle P \rangle$$

$$\frac{3\langle P \rangle V}{8\pi c r^2 h\nu} = \frac{3P \cdot \lambda^3}{8\pi \lambda \nu^2 r^2 h} = 1'62 \cdot 10^{15} \text{ fotoi}$$

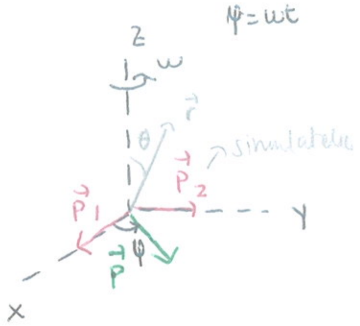
$$E_{max} = (E_\theta)_{max} = \frac{\langle [\ddot{p}] \rangle}{4\pi\varepsilon_0 r c^2} = \frac{\sqrt{6\pi\varepsilon_0 c^3 \langle P \rangle}}{4\pi\varepsilon_0 r c^2} = 0'67 \ V/m$$

hemen eradiazio dipolar elektrikoa dugu, ez da isotropoa eradiazioa

4.7.1 (Griffiths)

$p_0$ balioko dipolo biratzailea $\Rightarrow$ $\omega$ maiztasunarekin biratzen du haren norabidearekiko perpendikularra den ardatzaren inguruan. $\xrightarrow{} p$ $\omega$  $P_r$?

Hau kalkulatzeko bi dipolo oszilatzaileen konbinazioa aztertuko dugu, elkarren arteko angelu zuzena osatuz eta $\pi/2$-ko desfasearekin biratzen dutela onartuz.




$$\vec{p} = p_0 (\cos\omega t\,\hat{\imath} + \sin\omega t\,\hat{\jmath}) = \vec{p}_1 + \vec{p}_2 = p_0 \cos\omega t\,\hat{\imath} + p_0 \sin\omega t\,\hat{\jmath}$$

$$[\vec{p}_1] = p_0 \cos(\omega(t - r/c))\hat{\imath}, \quad [\vec{p}_2] = p_0 \sin(\omega(t-r/c))\hat{\jmath}$$

Dipolo bakoitzak irradiatzen duen eremua.

- $$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0 r^3 c^2} \vec{r} \times (\vec{r} \times [\ddot{\vec{p}}_1]) = -\frac{p_0\omega^2}{4\pi\varepsilon_0 r^3 c^2} \cos(\omega(t-r/c))\,\vec{r} \times (\vec{r} \times \hat{\imath})$$

$$[\ddot{\vec{p}}_1] = -p_0 \omega^2 \cos\omega(t-r/c)\hat{\imath}$$

- $$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0 r^3 c^2} \vec{r} \times (\vec{r} \times [\ddot{\vec{p}}_2]) = -\frac{p_0 \omega^2}{4\pi\varepsilon_0 r^3 c^2} \sin\omega(t-r/c)\,\vec{r} \times (\vec{r} \times \hat{\jmath})$$

$$[\ddot{\vec{p}}_2] = -p_0\omega^2 \sin\omega(t-r/c)\,\hat{\jmath}$$

- $$\vec{B}_1 = -\frac{\mu_0}{4\pi c r^2} \vec{r} \times [\ddot{\vec{p}}_1] = \frac{\mu_0 p_0 \omega^2 \cos\omega(t-r/c)}{4\pi c r^2} \vec{r} \times \hat{\imath}$$

- $$\vec{B}_2 = -\frac{\mu_0}{4\pi c r^2} \vec{r} \times [\ddot{\vec{p}}_2] = \frac{\mu_0 p_0 \omega^2 \sin\omega(t-r/c)}{4\pi c r^2} \vec{r} \times \hat{\jmath}$$

esferikoetan $\begin{cases} \hat{\imath} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{\jmath} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \end{cases}$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = -\frac{p_0\omega^2}{4\pi\varepsilon_0 r^3 c^2}((\cos\omega(t-r/c)\hat{\imath} + \sin\omega(t-r/c)\hat{\jmath}) \times \vec{r}) \times \vec{r} = \frac{-\mu_0 p_0 \omega^2}{4\pi\varepsilon_0 r c^2}(-\cos\theta(\cos\omega(t-r/c)\cos\phi +$$
$$\sin\omega(t-r/c)\sin\phi)\,\hat{\theta} + (\cos\omega(t-r/c)\sin\phi - \sin\omega(t-r/c)\cos\phi)\,\hat{\phi}) = \vec{E}_\theta + \vec{E}_\phi$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = -\frac{\mu_0 p_0 \omega^2}{4\pi c r^2}((\cos\omega(t-r/c)\hat{\imath} + \sin\omega(t-r/c)\hat{\jmath}) \times \vec{r} = +\frac{\mu_0 p \omega^2}{4\pi c r}((\cos\omega(t-r/c)\sin\phi +$$
$$-\sin\omega(t-r/c)\cos\phi)\,\hat{\theta} + \cos\theta(\cos\omega(t-r/c)\cos\phi + \sin\omega(t-r/c)\sin\phi)\,\hat{\phi}) = \vec{B}_\theta + \vec{B}_\phi$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2 (\cos^2\theta\,(\cos\omega(t-r/c)\cos\phi + \sin\omega(t-r/c)\sin\phi)^2 + (\cos\omega(t-r/c)\sin\phi +$$
$$-\sin\omega(t-r/c)\cos\phi)^2)\,\hat{r} \Rightarrow$$ (Konplexuekin ere egin ahal izango da)

$$\langle \vec{S} \rangle = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2 \Big[\cos^2\theta\big[\langle\cos^2\omega(t-r/c)\rangle\cos^2\phi + \langle\sin^2\omega(t-r/c)\rangle\sin^2\phi + \langle\cos\omega(t-r/c)\cdot$$

$$\sin\omega(\cancel{t-r/c})\rangle\sin\phi\cos\phi\big] + \big(\langle\cos^2\omega(t-r/c)\rangle\sin^2\phi + \langle\sin^2\omega(t-r/c)\rangle\cos^2\phi +$$

$$-\langle\cos\omega(t-r/c)\sin\cancel{\omega(t-r/c)}\rangle\sin\phi\cos\phi\big)\hat{r} = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2\Big[\frac{1}{2}\cos^2\theta(\cos^2\phi+\sin^2\phi) +$$

$$\frac{1}{2}\sin^2\phi+\cos^2\phi\Big]\hat{r} = \frac{\mu_0}{2c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2(\cos^2\theta+1)\hat{r}$$

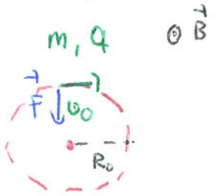
$$\langle P_r\rangle = \oint_{\text{Sfera}}\langle\vec{S}\rangle\cdot d\vec{s} = \frac{\mu_0}{2c}\left(\frac{p_0\omega^2}{4\pi}\right)^2\int_0^{\pi}\int_0^{2\pi}\frac{1+\cos^2\theta}{\cancel{r^2}}\cancel{r^2}\sin\theta\,d\phi\,d\theta = \frac{\mu_0}{2c}\left(\frac{p_0\omega^2}{4\pi}\right)^2\cdot$$

$$\int_0^{\pi}2\pi(\sin\theta+\sin\theta\cos^2\theta)\,d\theta = \frac{\mu_0}{c}\cdot\pi\left(\frac{p_0\omega^2}{4\pi}\right)^2\left[-\cos\theta-\frac{\cos^3\theta}{3}\right]\Big|_0^{\pi} = \frac{\mu_0\pi}{c}\left(\frac{p_0\omega^2}{4\pi}\right)^2\left(2+\frac{2}{3}\right)=$$

$$\frac{8}{3}\cdot\frac{\mu_0\pi}{c}\left(\frac{p_0\omega^2}{4\pi}\right)^2 = \frac{8\mu_0\pi p_0^2\omega^4}{3c\cdot 16\pi^2} = \frac{\mu_0 p_0^2\omega^4}{6\pi c}$$ (2 aldiz dipolo oszilatzaile bakar bateko erradiatzen duena)

4.8)

m masako eta q kargako partikula $\Rightarrow \vec{B}$ eremu magnetiko uniformean.




- $\vec{v}_0\perp\vec{B}$ , $T(t=0)=E_0=\frac{1}{2}mv_0^2 \to v_0=\sqrt{\frac{2E_0}{m}}$

- $F_{mag}=m\cdot a_n = m\cdot\frac{v_0^{\cancel{2}}}{R_0} = q\cancel{v_0}B \Rightarrow R_0=\frac{mv_0}{qB}=\frac{\sqrt{2E_0m}}{qB}$

irradiatutako duen potentzia

- $P_{irr} = -\frac{dE}{dt} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{q^2}{c^3}a^2$ (Larmor) ($v<<c$)

energia gutxitu irradazioaren ondorioz

azelerazioa beti da normala

$t$ aldiunean $\to T=\frac{1}{2}mv^2=E(t)\to v=\sqrt{\frac{2E(t)}{m}}$

$R=\frac{\sqrt{2mE(t)}}{q\cdot B}=\frac{m\cdot v}{qB}$ (txikitzen joan energia galtzean)

$a=\frac{v^2}{R}=\frac{v^{\cancel{2}}qB}{m\cancel{v}}=\sqrt{\frac{2E(t)}{m}}\frac{qB}{m}$

- $P_{irr}=\frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{q^2}{c^3}a^2 = \frac{1}{6\pi\varepsilon_0}\frac{q^2}{c^3}\cdot\frac{2E(t)}{m^3}q^2B^2 = \frac{q^4}{3\pi\varepsilon_0}\frac{E(t)B^2}{c^3m^3} = -\frac{dE}{dt}\Rightarrow$

$$\int_{E_0}^{E}\frac{dE}{E(t)} = -\frac{q^4}{3\pi\varepsilon_0}\frac{B^2}{c^3m^3}\int_0^t dt' = \underbrace{-\frac{q^4}{3\pi\varepsilon_0}\frac{B^2}{c^3m^3}}_{[\,]=1/t}t = \ln\frac{E(t)}{E_0}\Rightarrow E(t)=E_0e^{-\frac{q^4B^2}{3\pi\varepsilon_0c^3m^3}t}=E_0e^{-kt}$$

Beraz, $P_{irr} = \frac{q^4}{3\pi\varepsilon_0}\frac{B^2}{c^3m^3} E_0 e^{-\frac{q^4B^2}{3\pi\varepsilon_0 c^3m^3}t} = \frac{q^4}{3\pi\varepsilon_0}\frac{B^2}{c^3m^3} E_0 e^{-Kt}$

$K\cdot\tau = 1 \Rightarrow \quad E = \frac{E_0}{e} \quad ; \quad \tau = \frac{3\pi\varepsilon_0 c^3 m^3}{q^4B^2}$ (denbora karakteristikoa)

Bere ibilbidea espiral bat izango da, energia galduz joango denez bere erradioa txikituz joango delako, abiadurarekin batera: $R = \frac{\sqrt{2mE(t)}}{q\cdot B} = \frac{\sqrt{2mE_0}}{qB} e^{-\frac{q^4B^2 t}{6\pi\varepsilon_0 c^3m^3}}$

4.9)

atomo klasikoa

Atomo batek erradiazio dipolar elektrikoa igortzen du. $\Rightarrow \tau$?

elektroiaren azelerazioa

a) $P_r = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{e^2}{c^3} a^2 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{e^2}{c^3}\cdot w^2 \cdot \frac{2E(t)}{m} = -\frac{dE(t)}{dm} \Rightarrow$

$a = \frac{v^2}{r} = \frac{w^2r^2}{\cancel{r}} = w^2 r = w\cdot wr = w\cdot v$ *

$\int_{E_0}^{E(t)} \frac{dE(t)}{E(t)} = -\frac{e^2w^2}{\pi\varepsilon_0 3c^3 m}\int_0^t dt = -\frac{e^2w^2}{3\pi\varepsilon_0 c^3 m} t = \ln\frac{E(t)}{E_0} \Rightarrow E(t) = E_0 e^{-\frac{e^2w^2}{3\pi\varepsilon_0 c^3 m}t}$

$\tau = \frac{3\pi\varepsilon_0 c^3 m}{e^2w^2}$

* $F = \frac{+e^2}{4\pi\varepsilon_0 r^2} = m\cdot a = m\frac{v^2}{r} \rightarrow v^2 = \frac{+e^2}{4\pi\varepsilon_0 m r}$

$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r} = +\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0 r} = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r} = -\frac{1}{2}mv^2$

4.10.)

$e^-$ bat $\Delta V$ tentsiopean, X izpian hodi batean $\Rightarrow$ azeleratu eta $v$ abiadura lortu.

$$e\Delta V = \frac{1}{2}mv_0^2 \rightarrow v_0 = \sqrt{\frac{2e\Delta V}{m}}$$ (ez-erlatibista, balio numerikoak sartuz)

Anodoarekin talka egitean azelerazio uniformeaz $l$ distantzian erabat balaztatzen da.

- Guztiz geldtu: $v = v_0 - at = 0 \rightarrow a = \frac{v_0}{t}$

$$l = -\frac{1}{2}at^2 + v_0 t = -\frac{1}{2}v_0 t + v_0 t = \frac{1}{2}v_0 t \rightarrow t = \frac{2l}{v_0} = 1'5\cdot 10^{-12}\ s$$

$$a = \frac{v_0}{t} = \frac{v_0^2}{2l} = 8'844\cdot 10^{24}\ m/s^2$$

- Galdutako energia: $$P_{irr} = -\frac{dE}{dt} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{e^2}{c^3}a^2$$ (Larmor)

$$\Delta E = -\frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}a^2\int_0^t dt' = -\frac{1}{6\pi\varepsilon_0}\frac{e^2a^2t}{c^3} = -6'675\cdot 10^{-21}\ J = -4'2\cdot 10^{-2}\ eV$$

(Galdutako energia)

$$-\frac{\Delta E}{E_0} = -\frac{\Delta E}{q\Delta V} = \frac{0'04106\ eV}{50\ keV} = 0'8\cdot 10^{-6} \simeq 8\cdot 10^{-7} \simeq 1\ ppm$$ (part per million)

- %1 gutxienez erradiatu $\Rightarrow$ $-\frac{\Delta E}{E_0} = 0'01 \Rightarrow -\Delta E = 0'01 E_0 = 0'01\, q\Delta V =$

$$+P_{irr}\cdot\Delta t = P_{irr}\cdot\frac{2l}{v_0} \rightarrow a = \frac{v_0^2}{2l}$$

$$-\Delta E = P_{irr}\cdot\frac{2l}{v_0} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{e^2}{c^3}\cdot\left(\frac{v_0^2}{2l}\right)^2\cdot\frac{\cancel{2l}}{v_0} = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}\frac{v_0^3}{2l} = e\Delta V\cdot 0'01 \rightarrow$$

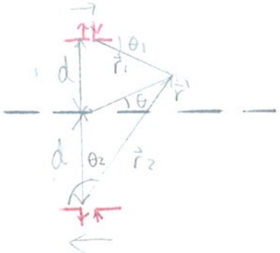
$$l = \frac{e^{\cancel{2}}v_0^3}{12\pi\varepsilon_0 c^3}\cdot\frac{1}{\cancel{e}\Delta V\cdot 0'01} = \frac{e\,v_0^3}{12\pi\varepsilon_0 c^3\Delta V\cdot 0'01} = \frac{e\cdot 2e\cancel{\Delta V}\sqrt{\frac{2e\Delta V}{m}}}{12\pi\varepsilon_0 m c^3\cancel{\Delta V}\cdot 0'01} = 8'28\cdot 10^{-14}\ m = 0'828\cdot 10^{-13}\ m = 0'828\cdot 10^{-3}\ Å$$

4.11.)

Antenak erode baten hurbiltasunean kokatu $\Rightarrow$ eroaleranitz irradiatutako energia islatu eta eremu osoa jatorrizkoa + islatutakoa da.

4.11.1 $r \simeq \lambda$, $L << \lambda$ $\Rightarrow$ erradiazio dipolar elektrikoa, induktiboko gunea. ?

Horizontalean:




$\vec{E} = \vec{E}_1 + \vec{E}_2$ , $\vec{B} = \vec{B}_1 + \vec{B}_2$

$r_1 = \sqrt{(r\sin\theta - d)^2 + r^2\cos^2\theta} = \sqrt{r^2 + d^2 - 2rd\sin\theta}$

$r_2 = \sqrt{(r\sin\theta + d)^2 + r^2\cos^2\theta} = \sqrt{r^2 + d^2 + 2rd\sin\theta}$

$$\begin{cases} \cos\theta_1 = \cos\theta \\ \sin\theta_1 = \dfrac{d - r\sin\theta}{r_1} \end{cases}$$

$$\begin{cases} \cos\theta_2 = -\cos\theta \\ \sin\theta_2 = \dfrac{r\sin\theta + d}{r_2} \end{cases}$$

4.12)

Uhin-erdiko antena: $L=\lambda/2$ $\qquad I(z,t)=I_0\cos\left(\frac{2\pi z}{\lambda}\right)\cos(\omega t')=I_0\cos\left(\frac{2\pi z}{\lambda}\right)\cos(\omega(t-R/c))$

$z\in(-\lambda/4,\lambda/4)$ $\qquad [\dot{p}]=I\cdot dz \quad , [\ddot{p}]=-\omega\, I_0\cos\left(\frac{2\pi z}{\lambda}\right)\sin\omega(t-R/c)\,dz$

Erradiazioko eremuak: $\vec{E}=\frac{[\ddot{p}]\sin\theta}{4\pi\varepsilon_0 R c^2}\hat{u}_\theta \rightarrow dE_\theta=\frac{-\sin\theta}{4\pi\varepsilon_0}\cdot\frac{\omega}{c^2}\frac{I_0\,dz}{R}\cos\left(\frac{2\pi z}{\lambda}\right)\sin\omega(t-R/c)$

$\vec{B}=\frac{\mu_0[\ddot{p}]\sin\theta}{4\pi R\cdot c}\hat{u}_\varphi \rightarrow dB_\varphi=-\frac{\mu_0\,\omega\, I_0\cos\left(\frac{2\pi z}{\lambda}\right)\sin\omega(t-R/c)\sin\theta}{4\pi c R}$

$u=\frac{2\pi z}{\lambda}\quad , z=\pm\lambda/4\rightarrow u=\pm\pi/2\qquad dz=\frac{\lambda}{2\pi}du$

$R=r-z\cos\theta=r-\frac{\lambda u}{2\pi}\cos\theta$

* $E_\theta=\int_{-\lambda/4}^{\lambda/4}dE_\theta=\int_{-\pi/2}^{\pi/2}-\frac{\sin\theta}{4\pi\varepsilon_0}\frac{\omega}{c^2}\frac{I_0}{R}\cos(u)\sin\omega(t-R/c)\frac{\lambda}{2\pi}du=-\frac{\sin\theta\,\omega}{4\pi\varepsilon_0c^2}\frac{I_0\cdot\lambda}{2\pi}\cdot$

$\int_{-\pi/2}^{\pi/2}\frac{\cos u}{R}\sin\left(\omega\left(t-\frac{r}{c}\right)+u\cdot\cos\theta\right)du=-\frac{\sin\theta\,\omega}{4\pi\varepsilon_0c^2}\frac{I_0\lambda}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{\cos u}{R}\left(\sin\omega\left(t-\frac{r}{c}\right)\cos(u\cos\theta)+\right.$

$\left.\sin(u\cos\theta)\cos\omega\left(t-\frac{r}{c}\right)\right)du=-\frac{\sin\theta\,\omega}{4\pi\varepsilon_0c^2}\frac{I_0\lambda}{2\pi}\cdot\frac{2\sin\omega(t-\frac{r}{c})}{r}\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}=$

$-\frac{\sin\theta\, I_0}{4\pi\varepsilon_0\cdot c}\,2\sin\omega\left(t-\frac{r}{c}\right)\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta\, r}$

* $B_\varphi=\int_{-\lambda/4}^{\lambda/4}dB_\varphi=-\frac{\mu_0\omega I_0\sin\theta\,\lambda}{c\,4\pi\,2\pi}\int_{-\pi/2}^{\pi/2}\frac{\cos u}{R}\sin\left(\omega\left(t-\frac{r}{c}\right)+u\cos\theta\right)du=-\frac{\mu_0}{4\pi}\frac{I_0\sin\theta}{r}\cdot$

$2\sin(\omega t-kr)\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$

* $\langle S\rangle=\frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)=\frac{1}{2}\cdot\frac{1}{\mu_0}\langle\vec{E}\times\vec{B}\rangle=\frac{1}{8\pi^2\varepsilon_0 c}\frac{I_0^2}{r^2}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$

* $\langle P_r\rangle=73'1\,\frac{I_0^2}{2}$

erradiazioko erresistentzia

4.13)

Dipolo erradiatzailea:

$$\text{norabidetasuna} = \frac{S_{max}}{P_r} = \frac{\frac{\cancel{[\ddot{p}]^2}}{16\pi^2}\cdot\frac{1}{\cancel{\varepsilon_0 c^3} r^2}}{\frac{1}{4\pi\cancel{\varepsilon_0}}\cdot\frac{2}{3}\frac{\cancel{[\ddot{p}]^2}}{\cancel{c^3}}} = \frac{6}{16\pi r^2} = \frac{3}{8\pi r^2}$$

Uhin-erdiko antena:

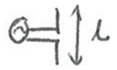
$$\text{norabidetasuna} = \frac{S_{max}}{P_r} = \frac{\frac{1}{8\pi^2\varepsilon_0 c}\frac{\cancel{I_0^2}}{r^2}}{73'1\,\frac{\cancel{I_0^2}}{2}} = \frac{2}{73'1\cdot 8\pi^2\varepsilon_0 c r^2}$$

4.14)

$L << \lambda$, bi dipoloak luzera berdinekoak $(L)$ eta korronte berdinaz zehokatzen dira.

$$I = I_0 e^{i\omega t}$$

Dipolo elektrikoa: $\langle P_{elek}\rangle = \frac{1}{4\pi\varepsilon_0}\frac{\omega^4}{c^3}\cdot\frac{1}{3}p_0^2 = \frac{1}{4\pi\varepsilon_0}\frac{\omega^4}{c^3}\cdot\frac{1}{3}\cdot\frac{l^2 I_0^2}{\omega^2}$

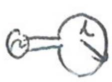


$p = p_0 e^{i\omega t} = q\cdot l$

$\ddot{p} = -\omega^2 p_0 e^{i\omega t} = I_0 e^{i\omega t}\cdot l\,(-\omega)$

$I_0 = \frac{\omega p_0}{l} \rightarrow p_0 = \frac{l I_0}{\omega}$

Dipolo magnetikoa:



$$\langle P_{mag}\rangle = \frac{\mu_0}{4\pi}\cdot\frac{1}{3}\frac{m_0^2\omega^4}{c^3} = \frac{\mu_0}{4\pi}\cdot\frac{1}{3}\frac{I_0^2\pi^2 l^4\omega^4}{c^3}$$

$m_0 = I_0\pi l^2$

$$\frac{\langle P_{elek}\rangle}{\langle P_{mag}\rangle} = \frac{\frac{1}{\cancel{4\pi}\varepsilon_0}\frac{\omega^2}{\cancel{c^3}}\cdot\cancel{\frac{1}{3}}\,l^2\cancel{I_0^2}}{\frac{\mu_0}{\cancel{4}}\cdot\cancel{\frac{1}{3}}\frac{\cancel{I_0^2\pi}\,l^4\omega^4}{\cancel{c^3}}} = \frac{1}{\varepsilon_0\mu_0}\cdot\frac{1}{\omega^2}\cdot\frac{1}{l^2} = \frac{c^2}{\omega^2 l^2} = \frac{c^2}{4\pi^2\nu^2 l^2}$$

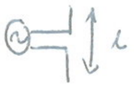
a) $\frac{\langle P_{elek}\rangle}{\langle P_{mag}\rangle} = 2'27\cdot 10^5$

b) $\frac{\langle P_{elek}\rangle}{\langle P_{mag}\rangle} = 2'3\cdot 10^5$ (neurriak ~ aldatu)

c) $\frac{\langle P_{elek}\rangle}{\langle P_{mag}\rangle} = 22'79$

4.15)

Dipolo elektrikoa:



$$\langle P_{elek}\rangle = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{3}\frac{p_0^2\omega^4}{c^3} \overset{p_0\omega = I_0\cdot l}{=} \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{3}\frac{I_0^2 l^2\omega^2}{c^3} \overset{\omega^2/c^2 = \left(\frac{2\pi}{\lambda}\right)^2}{=} \frac{1}{\cancel{4\pi}\varepsilon_0}\cdot\frac{1}{3}\frac{\cancel{4}\pi^2}{c\lambda^2} l^2 I_0^2 =$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\pi}{3}\left(\frac{l}{\lambda}\right)^2 I_0^2 = \underbrace{\frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\left(\frac{l}{\lambda}\right)^2}_{R_{elek}}\cdot\frac{I_0^2}{2}$$

(erradiaziozko erresistentzia)

Dipolo magnetikoa:

$$\langle P_{mag}\rangle = \frac{\mu_0}{4\pi}\cdot\frac{1}{3}\frac{m_0^2\omega^4}{c^3} \overset{m_0 = I_0\pi l^2}{=} \frac{\mu_0}{4\cancel{\pi}}\cdot\frac{1}{3}\frac{I_0^2\pi^{\cancel{2}} l^4\omega^4}{c^3} = \frac{\mu_0}{12}\frac{l^4\omega^4\pi}{c^3} I_0^2 \overset{\omega^4/c^4 = \left(\frac{2\pi}{\lambda}\right)^4}{=}$$

$$\frac{\mu_0}{12}\frac{l^4\, c\pi 16\pi^4}{\lambda^4} I_0^2 = \sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\frac{4}{3}\pi^5\left(\frac{l}{\lambda}\right)^4 I_0^2 = \underbrace{\frac{8}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\pi^5\left(\frac{l}{\lambda}\right)^4}_{R_{mag}}\frac{I_0^2}{2}$$

(erradiaziozko erresistentzia)

$$\frac{R_{elek}}{R_{mag}} = \frac{\frac{2\pi}{\cancel{3}}\cancel{\sqrt{\frac{\mu_0}{\varepsilon_0}}}\cancel{\left(\frac{l}{\lambda}\right)^2}}{\frac{8}{\cancel{3}}\cancel{\sqrt{\frac{\mu_0}{\varepsilon_0}}}\pi^5\left(\frac{l}{\lambda}\right)^{\cancel{4}\,2}} = \frac{1}{4\pi^4}\cdot\left(\frac{\lambda}{l}\right)^2$$

4.16)

$\omega$ maiztasuneko uhin elektromagnetiko monokromatiko bat detektatu nahi da dipolo bat edo begizta zirkular bat erabiliz antena hartzaile modura.

Uhina z norabidean hedatu: $\vec{E} = E\hat{\imath}$, $\vec{B} = B\hat{\jmath}$, $\vec{S} = S\hat{k}$ $(L \ll \lambda)$

Dipoloa: Uhin elektromagnetikoaren osagai tangentziala neurtzen du $\Rightarrow$ induzituko seinalea ahalik eta handiena izateko $\vec{E}$ eta $\vec{l}$ paraleloak izan behar dira $\Rightarrow$ dipoloaren norabidea, $\vec{p} = p\hat{\imath}$ izan behar da.

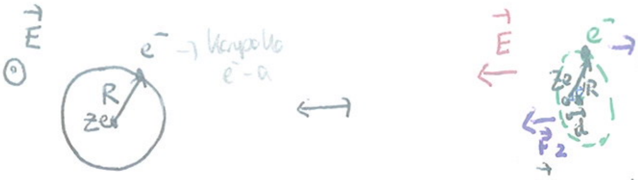
Begizta zirkularra: Espazioko osagai perpendikularra baino ez du neurtzen $\Rightarrow$ induzituko seinalea ahalik eta handiena izateko begizta $\vec{B}$-ren perpendikularra izan behar da, hau da, $\vec{S} = S\hat{\jmath}$


(azalera)

# 5. GAIA: MATERIAREN
# TEORIA ELEKTROMAGNETIKOA

16-12-08

**5.1.)** $\alpha$? $\langle \vec{p}_m \rangle = \alpha \vec{E}$ ; $d \Rightarrow$ elektroiaren eta nukleoaren arteko desplazamendu erlatiboa




$R \gg d$

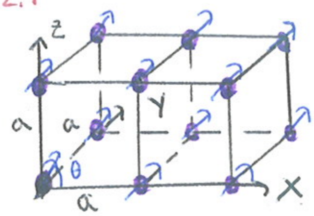
$\vec{p}_m = d \cdot Ze$

Orekan nukleoaren eta elektroiaren arteko erakarpen indarraren osagai perpendikularra eta eremuak elektroian gainean eragindakoa berdinak izan behar dira.

$$F_1 = eE\,,\quad F_2 = \frac{Ze\cdot e}{4\pi\varepsilon_0 R^2}\cdot\overset{\cos\theta}{\overset{\shortparallel}{\frac{d}{R}}} \;\leftrightarrow\; \cancel{e}E = \frac{Ze^{\cancel{2}}\cdot d}{4\pi\varepsilon_0 R^3} \Rightarrow Zed = p_m = \underbrace{4\pi\varepsilon_0 R^3}_{\alpha} E$$

Beraz $\alpha = 4\pi\varepsilon_0 R^3$ da.

**5.2.)**




$\vec{p}_0 = p_0\cos\theta\,\hat{\imath} + p_0\sin\theta\,\hat{k} = p_0(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{k})$

Molekulen sare kubikoa $\Rightarrow$ molekula bakoitzak 6 molekula ditu inguruan $a$ distantziara, hamabi $a\sqrt{2}$ distantziara... Molekula guztiek $p_0$ momentu dipolar elektriko iraunkorra dute, guztiek norabidea berdina izanik. Molekula batean inguruko molekula hurbilenek sortuko eremua?

Molekula hurbilenek sortzen duten eremua molekula batean $\Rightarrow$ $d = a$ (6 molekula)

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3(\vec{p}_0\cdot\vec{r})\cdot\vec{r}}{r^2} - \vec{p}_0\right) = \frac{1}{4\pi\varepsilon_0 r^3}\left(3p_0\cos\theta\,\hat{\imath} - p_0(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{k})\right) = \frac{p_0(2\cos\theta\,\hat{\imath} - \sin\theta\,\hat{k})}{4\pi\varepsilon_0 r^3}$$

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3p_0\cos\theta(-\cancel{r})\cdot(-\cancel{r}\hat{\imath})}{\cancel{r^2}} - p_0(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{k})\right) = \vec{E}_1$$

$$\vec{E}_3 = \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3p_0\sin\theta\cdot\cancel{r}\cdot\cancel{r}\hat{k}}{\cancel{r^2}} - p_0(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{k})\right) = \frac{p_0(2\sin\theta\,\hat{k} - \cos\theta\,\hat{\imath})}{4\pi\varepsilon_0 r^3} = \vec{E}_4$$

$$\vec{E}_5 = \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3\cancel{(\vec{p}_0\cdot r\hat{\jmath})}\,r\hat{\jmath}}{r^2} - \vec{p}_0\right) = \frac{-\vec{p}_0}{4\pi\varepsilon_0 r^3} = \vec{E}_6 = \frac{-p_0(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{k})}{4\pi\varepsilon_0 r^3}$$

$$\vec{E}_m = \sum_{i=1}^{6} \vec{E}_i = 2\vec{E}_1 + 2\vec{E}_3 + 2\vec{E}_6 = \frac{2p_0}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{i} - \sin\theta\,\hat{k} + 2\sin\theta\,\hat{k} - \cos\theta\,\hat{i} - \cos\theta\,\hat{i} - \sin\theta\,\hat{k}\right) = 0$$

$r = \sqrt{2}a$ distantziara daudenak $\Rightarrow$ 12 molekula.

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\cdot\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) - \vec{p}_0\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{3\cos\theta}{\sqrt{2}}\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) - \cos\theta\,\hat{i} - \sin\theta\,\hat{k}\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{3}{2}\cos\theta(\hat{i}+\hat{j}) +\right.$$

$$\left. - \cos\theta\,\hat{i} - \sin\theta\,\hat{k}\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{\cos\theta}{2}(\hat{i}+3\hat{j}) - \sin\theta\,\hat{k}\right)$$

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\left(\frac{\hat{i}-\hat{j}}{\sqrt{2}}\right)\left(\frac{\hat{i}-\hat{j}}{\sqrt{2}}\right) - \vec{p}_0\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{3\cos\theta}{2}(\hat{i}-\hat{j}) - \cos\theta\,\hat{i} - \sin\theta\,\hat{k}\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{\cos\theta}{2}(\hat{i}-3\hat{j}) - \sin\theta\,\hat{k}\right)$$

$$\vec{E}_3 = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\cdot\left(\frac{-\hat{i}-\hat{j}}{\sqrt{2}}\right)\left(\frac{-\hat{i}-\hat{j}}{\sqrt{2}}\right) - \vec{p}_0\right) = \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3\vec{p}_0\cdot(\hat{i}+\hat{j})}{2}(\hat{i}+\hat{j}) - \vec{p}_0\right) = \vec{E}_1$$

$$\vec{E}_4 = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\cdot\left(\frac{-\hat{i}+\hat{j}}{\sqrt{2}}\right)\left(\frac{-\hat{i}+\hat{j}}{\sqrt{2}}\right) - \vec{p}_0\right) = \vec{E}_2$$

$$\vec{E}_5 = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\cdot\left(\frac{\hat{i}+\hat{k}}{\sqrt{2}}\right)\left(\frac{\hat{i}+\hat{k}}{\sqrt{2}}\right) - \vec{p}_0\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{3}{2}(\cos\theta+\sin\theta)(\hat{i}+\hat{k}) - \sin\theta\,\hat{k} - \cos\theta\,\hat{i}\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\cdot$$

$$\left(\hat{i}\left(\frac{1}{2}\cos\theta + \frac{3}{2}\sin\theta\right) + \hat{k}\left(\frac{3}{2}\cos\theta + \frac{1}{2}\sin\theta\right)\right)$$

$$\vec{E}_6 = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\cdot\left(\frac{-\hat{i}-\hat{k}}{\sqrt{2}}\right)\left(\frac{-\hat{i}-\hat{k}}{\sqrt{2}}\right) - \vec{p}_0\right) = \vec{E}_5$$

$$\vec{E}_7 = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\left(\frac{\hat{i}-\hat{k}}{\sqrt{2}}\right)\left(\frac{\hat{i}-\hat{k}}{\sqrt{2}}\right) - \vec{p}_0\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{3}{2}(\cos\theta-\sin\theta)(\hat{i}-\hat{k}) - \cos\theta\,\hat{i} - \sin\theta\,\hat{k}\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\cdot$$

$$\left(\frac{\hat{i}}{2}(\cos\theta - 3\sin\theta) + \frac{\hat{k}}{2}(\sin\theta - 3\cos\theta)\right) = \vec{E}_8$$

$$\vec{E}_9 = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\cdot\left(\frac{\hat{k}+\hat{j}}{\sqrt{2}}\right)\left(\frac{\hat{k}+\hat{j}}{\sqrt{2}}\right) - \vec{p}_0\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{3}{2}(\sin\theta)(\hat{k}+\hat{j}) - \cos\theta\,\hat{i} - \sin\theta\,\hat{k}\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\hat{j}\,\frac{3}{2}\sin\theta +\right.$$

$$\left.\frac{\hat{k}}{2}\sin\theta - \cos\theta\,\hat{i}\right) = \vec{E}_{10} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\left(\frac{-\hat{k}-\hat{j}}{\sqrt{2}}\right)\left(\frac{-\hat{k}-\hat{j}}{\sqrt{2}}\right) - \vec{p}_0\right)$$

$$\vec{E}_{11} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\left(\frac{-\hat{k}+\hat{j}}{\sqrt{2}}\right)\left(\frac{-\hat{k}+\hat{j}}{\sqrt{2}}\right) - \vec{p}_0\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(\frac{3}{2}(-\sin\theta)(-\hat{k}+\hat{j}) - \cos\theta\,\hat{i} - \sin\theta\,\hat{k}\right) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(-\cos\theta\,\hat{i} + \frac{\sin\theta}{2}(\hat{k}-3\hat{j})\right) =$$

$$\vec{E}_{12} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3\vec{p}_0\cdot\left(\frac{-\hat{j}+\hat{k}}{\sqrt{2}}\right)\cdot\left(\frac{-\hat{j}+\hat{k}}{\sqrt{2}}\right) - \vec{p}_0\right)$$

$$\vec{E}_m = \sum_{i=1}^{12}\vec{E}_i = 2\vec{E}_1 + 2\vec{E}_2 + 2\vec{E}_5 + 2\vec{E}_7 + 2\vec{E}_9 + 2\vec{E}_{11} = \frac{2p_0}{4\pi\varepsilon_0 r^3}\left(\frac{\cos\theta}{\cancel{2}}(\hat{i}+3\cancel{\hat{j}}) - \sin\theta\,\hat{k} + \frac{\cos\theta}{\cancel{2}}(\hat{i}-3\cancel{\hat{j}}) - \sin\theta\,\hat{k} +\right.$$

$\frac{\hat{i}}{2}\cancel{(\cos\theta + 3\sin\theta)} + \frac{\hat{k}}{2}(3\cos\theta + \sin\theta) + \frac{\hat{i}}{2}\cancel{(\cos\theta - 3\sin\theta)} + \frac{\hat{k}}{2}(\sin\theta - 3\cos\theta) + \frac{3}{2}\sin\theta\,\hat{j} + \frac{\hat{k}}{2}\sin\theta - \cancel{\cos\theta\,\hat{i}} +$

$- \cancel{\cos\theta\,\hat{i}} + \frac{\sin\theta}{2}\hat{k} - \frac{3\hat{j}}{2}\sin\theta) = \frac{2p_0}{4\pi\varepsilon_0 r^3}\left(-2\sin\theta\,\hat{k} + \hat{k}\sin\theta + \frac{3}{2}\cancel{\sin\theta}\,\hat{j} + \hat{k}\sin\theta - \frac{3}{2}\cancel{\sin\theta\,\hat{j}}\right) = 0$

?

5.3)

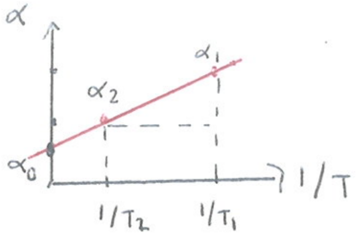
$CO_2$, $NH_3$

| T (K) | $\varepsilon_r$ ($CO_2$) | $\varepsilon_r$ ($NH_3$) |
|---|---|---|
| 273 | 1'000988 | 1'00834 |
| 373 | 1'000723 | 1'00487 |

(p = 1 atm)

Molekula molarrak dira, beraz momentu dipolarra dute ⇒ orientaziozko polarizibilitatea

$$\alpha_{orient} = \frac{p_0^2}{3k_B T} = \frac{p_0^2}{3k_B}\cdot\frac{1}{T}$$




? $\alpha = \alpha_0 + \frac{p_0^2}{3k_B}\cdot\frac{1}{T}$

• $CO_2 \Rightarrow \alpha(T_1) = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = 3'258\cdot10^{-40}$ F/m (Clausius-Mossotti)

$$N = \frac{1960\,\cancel{g}}{m^3}\cdot\frac{1\,\cancel{mol\,CO_2}}{44\,g}\cdot\frac{N_A\text{ molek.}}{1\,\cancel{mol\,CO_2}} = 2'68\cdot10^{25}\text{ molekula}/m^3$$

$$\alpha(T_2) = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = 2'385\cdot10^{-40}\text{ F/m}$$

$$\alpha(T) = \overbrace{\alpha_0}^{\text{elektronikoa}} + \alpha_{orient} = 4\pi\varepsilon_0 R^3 + \overbrace{\frac{p_0^2}{3k_B}\cdot\frac{1}{T}}^{\text{balio teorikoa}} = \alpha(T)$$

$$\alpha(T_1) - \alpha(T_2) = \frac{p_0^2}{3k_B}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \leftrightarrow \sqrt{\frac{3k_B(\alpha(T_1) - \alpha(T_2))}{\left(\frac{1}{T_1} - \frac{1}{T_2}\right)}} = p_0 = 1'9188\cdot10^{-30}\text{ C·m}$$

$$4\pi\varepsilon_0 R^3 = \alpha(T_1) - \frac{p_0^2}{3k_B}\cdot\frac{1}{T_1} \Rightarrow R = \sqrt[3]{\frac{1}{4\pi\varepsilon_0}\left(\alpha(T_1) - \frac{p_0^2}{3k_B}\cdot\frac{1}{T_1}\right)} = 7'17\cdot10^{-12}\text{ m}$$

• $NH_3 \Rightarrow N = \frac{730\,\cancel{g}}{m^3}\cdot\frac{1\,\cancel{mol\,NH_3}}{17\,g}\cdot\frac{N_A\text{ molekula}}{1\,\cancel{mol\,NH_3}} = 2'586\cdot10^{25}$ molekula/$m^3$

$\alpha(T_1) = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = 2'8496\cdot10^{-39}$ F/m    $\alpha(T_2) = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = 1'6639\cdot10^{-39}$ F/m

$$\alpha(T) = \alpha_0 + \alpha_{orient} = 4\pi\varepsilon_0 R^3 + \frac{p_0^2}{3k_B}\cdot\frac{1}{T}$$

$$\alpha(T_1) - \alpha(T_2) = \frac{p_0^2}{3k_B}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \Rightarrow p_0 = \sqrt{\frac{3k_B(\alpha(T_1)-\alpha(T_2))}{\left(\frac{1}{T_1}-\frac{1}{T_2}\right)}} = 7'069\cdot 10^{-30}\ C\cdot m \quad , \quad 3'247\cdot 10^{-29}\ C\cdot m$$

$$R = \sqrt[3]{\frac{1}{4\pi\varepsilon_0}\left(\alpha(T_1) - \frac{p_0^2}{3k_B}\frac{1}{T_1}\right)} \quad , \quad 1'9339\cdot 10^{-10}\ m$$

5.4)

Clausius-Mossotti: $\alpha = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)$ $\qquad \alpha = \alpha_{orient.} + \alpha_{ion.} + \alpha_{elek.}$

$f$ altuetan bakarrik $f >>$

Maiztasun handietan ($f > f_{opt}$ ($10^{14}$ Hz)) $\alpha = \alpha_{elek}$ bakarrik eta $n \simeq \sqrt{\varepsilon_r}$

tarte optikoa

$$\alpha = 4\pi\varepsilon_0 R^3 = \alpha_{elek} = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = \frac{3\varepsilon_0}{N}\left(\frac{n^2-1}{n^2+2}\right) \Rightarrow \frac{n^2-1}{n^2+2} = \frac{N}{3\cancel{\varepsilon_0}}\cdot 4\pi\cancel{\varepsilon_0} R^3 =$$

$$N\frac{4\pi}{3}R^3 = \frac{N_A\cdot\rho}{P_m}\cdot\frac{4}{3}\pi R^3 = N_A\cdot\frac{\rho\cdot V}{P_m} \Rightarrow \frac{1}{\rho}\left(\frac{n^2-1}{n^2+2}\right) = N_A\cdot\frac{V}{P_m}$$

Pisu molarra

5.5)

$\varepsilon_s = 78'4$, $T = 298'15\ K \Rightarrow$ jaitsi $\varepsilon_r = 20$ balora $\qquad f = 40\ GHz = 40\cdot 10^9\ Hz$

Maiztasun optikoko (altuko) uraren konstante dielektrikoa, $\varepsilon_r = \varepsilon_\infty = 5 \Rightarrow \tau$ ?

$\varepsilon_\infty \Rightarrow \alpha_{elek} = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_\infty - 1}{\varepsilon_\infty + 2}\right)$ ; Estatikoa $\Rightarrow$ $\alpha_{orient}(0) = \frac{p_0^2}{3k_BT} = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_s - 1}{\varepsilon_s + 2}\right) \rightarrow$

$$N = \frac{1000\ \cancel{g}}{m^3}\cdot\frac{1\ \cancel{mol\ H_2O}}{18\ g}\cdot\frac{N_A\ molekula}{1\ \cancel{mol\ H_2O}} = 3'345\cdot 10^{25}\ molekula/m^3$$

$\alpha(0) = 7'639\cdot 10^{-37}\ F/m$ ; $\alpha^*(\omega) = \alpha(\omega) + i\alpha'(\omega) = \frac{\alpha(0)}{1+\omega^2\tau^2}(1 + i\omega\tau) = \frac{\alpha(0)}{1 - i\omega\tau}$

$$\alpha(\omega) = \frac{\alpha(0)}{1+\omega^2\tau^2} = \frac{(\alpha_{elek} + \alpha_{orient})}{1+\omega^2\tau^2} = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)$$

$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + i\omega\tau}$ $\qquad \tau = 7'2\ ps$

5.6.)

$p_0 = 6'16 \cdot 10^{-30}$ C·m (Ur molekulen momentu dipolar iraunkorra) $\Rightarrow T_c$ ? → ferroelektrikoak

$N_{H_2O} = \frac{1000 g}{m^3} \cdot \frac{1 mol H_2O}{18 g} \cdot \frac{N_A \text{ molekula}}{1 mol H_2O} = 3'345 \cdot 10^{25}$ molekula $/m^3$

Ura polarra $\Rightarrow$ $\alpha \simeq \alpha_{orient}$ (→ bizarkoa) $= \frac{p_0^2}{3 k_B T}$

$\frac{N\alpha}{3\varepsilon_0} = 1 \rightarrow \alpha = \frac{3\varepsilon_0}{N}$

$\frac{p_0^2}{3k_B T} = \frac{3\varepsilon_0}{N} \Rightarrow T = \frac{N p_0^2}{9\varepsilon_0 k_B} = 1155$ K

(Tenperatura horretan ura gas egoeran dago $\Rightarrow$ ez da ferroelektrikoa)

5.7.)

Cu-ko haria $\Rightarrow$ $\langle v \rangle = v_A$ ? (Jito-abiadura) $J = 10 A/mm^2$ ; 1 $\bar{e}$ aske / atomo Cu

- $J = N q v_A$ ; $N = \frac{8'93 \cancel{g}}{\cancel{cm^3}} \cdot \frac{1 \cancel{cm^3}}{10^{-6} m^3} \cdot \frac{1 \cancel{mol}}{63'5 g} \cdot \frac{N_A \cancel{\text{atomo Cu}}}{1 \cancel{mol}} \cdot \frac{1\ \bar{e} \text{ aske}}{1 \cancel{\text{atomo Cu}}} = 8'468 \cdot 10^{28}\ \bar{e}/m^3$

- $v_A = \frac{J}{Nq} = \frac{10 A/mm^2}{8'468 \cdot 10^{18} \bar{e}/cm^3 \cdot e} = \frac{10 A \cdot 10^6/m^2}{e \cdot N} = 7'38 \cdot 10^{-4}\ m/s \simeq 0'738\ mm/s$

$T = 300 K \Rightarrow v_T$ ? $\quad v_T = \sqrt{\frac{3k_B T}{m}} = 1'168 \cdot 10^5\ m/s \gg v_A$

5.8.)

Cu, $T = 273 K$, $\bar{e}$ askeen gasaren presioa ?

Gasa $\Rightarrow$ $pV = nRT \Leftrightarrow p = \frac{nRT}{V} = \frac{nRT}{V} \cdot \frac{N_A}{N_A} = \frac{RT}{N_A} N = 3'147 \cdot 10^6$ atm $\simeq 31'47$ Pa

$N = 8'468 \cdot 10^{28}\ \bar{e}/m^3$

5.9.)

$\sigma = 5'9 \cdot 10^7 (\Omega m)^{-1}$, $T \simeq 300 K$ (inguruko tenperatura), $\tau$ ? $\lambda$ ? → Batez-besteko ibilbide askea. $\omega_p$, $\gamma = 1/\tau$ ?

$\sigma = \frac{Nq^2}{m}\tau \Rightarrow \tau = \frac{m\sigma}{Nq^2} = \frac{m_e \cdot \sigma_{Cu}}{N \cdot e^2} = 2'48 \cdot 10^{-14}$ s $\quad \lambda = v_T \tau = 2'898 \cdot 10^{-9}\ m \simeq 29$ Å

($N = 8'45 \cdot 10^{28}$ atomo$/m^3$) $\quad$ (5.7 ariketatik)

$$\omega_p^2 = \frac{Nq^2}{\varepsilon_0 m} = \frac{\sigma(0)}{\varepsilon_0}\gamma \Rightarrow \quad \omega_p = \sqrt{\frac{\sigma(0)\gamma}{\varepsilon_0}} = \sqrt{\frac{\sigma(0)}{\tau\varepsilon_0}} = 1'639 \cdot 10^{16} \text{ rad/s}$$

$$\frac{1}{\gamma} = \frac{\sigma(0)}{\varepsilon_0 \omega_p^2} = \tau \Rightarrow \quad \gamma = \frac{\varepsilon_0 \omega_p^2}{\sigma(0)} = 4'032 \cdot 10^{13} \text{ s}^{-1} \quad << \omega \qquad (\text{ircole ez})$$

5.10)

a) $\rho_{Ge}$? $T = 300K$, $\mu_e = 3900 \text{ cm}^2/Vs$ (elektroiak) $\mu_z = 1900 \text{ cm}^2/Vs$ (zuloak), $n = 2'5 \cdot 10^{13}$ karga-eramaile/$cm^3$

Erabiltzalea $\Rightarrow \sigma = nq(\mu_e + \mu_z) = \frac{1}{\rho} \Leftrightarrow \rho = \frac{1}{ne(\mu_e+\mu_z)} = 43'1 \ \Omega cm = 0'431 \ \Omega m$

$* \ n = n_0 e^{-E_g/2k_BT} \Leftrightarrow \ln\frac{n}{n_0} = -\frac{E_g}{2k_BT} = \quad E_g = 2k_BT \ln\frac{n_0}{n}$

b) $\rho_{GaAs}$? $T = 300K$, $\mu_e = 8500 \text{ cm}^2/V\cdot s$ $\quad \mu_z = 400 \text{ cm}^2/Vs$, $n = 2\cdot 10^6$ karga-eramaile/$cm^3$

Erabiltzalea $\Rightarrow \sigma = nq(\mu_e + \mu_z) = \frac{1}{\rho} \Leftrightarrow \rho = \frac{1}{ne(\mu_e+\mu_z)} = 3'51 \cdot 10^8 \ \Omega \cdot cm = 3'51 \cdot 10^6 \ \Omega m$

5.11)

$$\chi_{dmag} = -\frac{\mu_0 N e^2 Z}{6m}\langle R^2\rangle \Rightarrow \text{molara} \Rightarrow \chi_{dia}^{mol} = \frac{\chi_{dia}}{N} \overset{V/mol}{N_A} = -\frac{\mu_0 \cancel{N} e^2 Z \langle R^2\rangle N_A}{\cancel{N}\cdot 6m} =$$

$$-\frac{\mu_0 e^2 Z\langle R^2\rangle N_A}{6m} \Rightarrow \text{Å-etan eman} \Rightarrow \chi_{dia}^{mol} = -3'548\cdot 10^9 \cdot Z\langle R^2_{(m)}\rangle \ (m^3/mol) \Rightarrow \text{Å} \Rightarrow$$

$$\chi_{dia}^{mol} = \underbrace{-3'54\cdot 10^{-11}}_{m^3/(\text{Å})^2} Z\langle R^2(\text{Å})\rangle \ (m^3/mol)$$

? ergatik?

$$Ar = \begin{cases} Z = 18 \\ R = 1'88 \text{ Å} \end{cases} \Rightarrow \chi_{dia}^{mol} = -2'25\cdot 10^{-9} \ m^3/mol > C_{Ar} = -2'44 \cdot 10^{-10} \ m^3/mol$$

5.12)

$\gamma$? $H_m = \gamma M$, $T_C = 1044K$ $\quad m_0 = 2'2 \ \mu_B$ $\quad M_m = 55'85$ g/mol $\quad \rho = 7'82$ g/cm$^3$

$$T_C = \gamma\frac{M_0 \mu_0 m_0}{3K_B} = \underbrace{\frac{N\mu_0 m_0^2}{3K_B}}_{C}\gamma = C\gamma \Leftrightarrow \gamma = \frac{T_C}{C} = \frac{1044K\cdot 3\cdot K_B}{N\mu_0 m_0^2} = 980'75 >> 1/3$$

$$N = \frac{7'82 \ \cancel{g}}{10^{-6} m^3}\cdot\frac{1 \ \cancel{mol}}{55'85 \ \cancel{g}}\cdot\frac{N_A \text{ atomo}}{1 \ \cancel{mol}} = 8'43\cdot 10^{28} \text{ atomo}/m^3$$

cdc   $M \geq N m_0$

5.13.)

$B_m$ ? $\quad \mu_0 M_s = 2'2\ T$ (→ asdarrakiko ?) $\quad H_m = \gamma M \Rightarrow B_m = \mu_0 H_m = \mu_0 \gamma M = \gamma \cdot 2'2\ T = 2'16\,10^3\ T = 2160\ T$

5.14.)

Inguruna ferromagnetikoa → $\chi(T)$ $\quad T \gg T_C$

$\chi = M/H_0$ $\quad$ ($H_0$ ⇒ aplikaturiko eremu magnetikoa)

$T \gg T_C$ ⇒ paramagnetiko moduan jokatu.

$$\chi_{pm} = \frac{M}{H_0} = \frac{M}{H - H_m} = \frac{1}{\frac{H}{M} - \frac{\gamma M}{M}} = \frac{1}{\frac{T}{C} - \gamma} = \frac{C}{T - C\gamma} = \frac{C}{T - \theta} = \frac{C}{T - T_C}$$

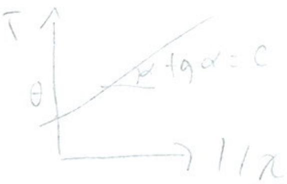
$\left( C = \frac{N \mu_0 m_0^2}{3 k_B} \right)$ $\quad$ $(T_C = \gamma \cdot C)$

5.15.)

$CePd_2Si$ ⇒

| $T(K)$ | $10^2\,\chi\ (\mu_B/T)$ | $1/\chi\ (\mu_B/T)$ |
|---|---|---|
| 75 | 1'612 | 62'03 |
| 100 | 1'265 | 79'05 |
| 175 | 0'735 | 136'05 |
| 200 | 0'641 | 156'01 |
| 250 | 0'521 | 191'93 |
| 300 | 0'441 | 226'75 |

$T = \theta + \frac{C}{\chi}$




Erregresio lineala egin ⇒ $T(1/\chi)$ ? $\quad T(1/\chi) = -8'68_{\nearrow (K)} + 1'352_{\nearrow (K \cdot \mu_B / T)} \cdot 1/\chi$

Curie-Weiss ⟷ $T - T_C = C \cdot \frac{1}{\chi}$ ⟷ $T(1/\chi) = T_C + \frac{C}{\chi}$ $\quad C = 1'352\ (K \cdot \mu_B)$

$m_0$ ? $\quad C = 1'3521 = \frac{N \mu_0 m_0^2}{3 k_B} \Rightarrow m_0 = \sqrt{1'3521 \cdot 3 k_B \cdot \frac{1}{N \mu_0}} = 2'458\,\mu_B$
?

$N_{Ce} = \frac{6'689 \cdot 10^3\ g}{m^3} \cdot \frac{1\ mol}{140'12\ g} \cdot \frac{N_A\ atomo}{1\ mol} = 2'87 \cdot 10^{28}$ atomo/$m^3$
?

5.16.)

$\vec{H}_0$, $\vec{M} = \vec{H}_m\ 1/\gamma \rightarrow \vec{H}_0 \cdot \vec{M} = (\vec{H} - \vec{H}_m) \cdot \vec{M} = (\vec{H} - \vec{H}_m) \cdot \frac{\vec{H}_m}{\gamma} = \frac{1}{\gamma} (\vec{H} \cdot \vec{H}_m - H_m^2)$

# ELEKTROMAGNETISMOA II:

# 6. GAIA: ERLATIBITATEA eta ELEKTROMAGNETISMOA

16-12-15

6.1.)

a) Uhin ekuazioa: $\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$

Galileoren transformazioak $\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$

- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ ; $\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\partial}{\partial x'}$; $\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x'^2}$

- $\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}$ ; $\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$ ; $\underset{\downarrow\ t(t',x')}{\frac{\partial}{\partial t}} = \frac{\partial}{\partial x'}\cdot\frac{\partial x'}{\partial t} + \frac{\partial}{\partial t'}\cdot\frac{\partial t'}{\partial t} = -v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}$

- $\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y'^2}$ ; $\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z'^2}$ ; $\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right) = \left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)\left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right) =$

$$v^2\frac{\partial^2}{\partial x'^2} - 2v\frac{\partial^2}{\partial x'\partial t'} + \frac{\partial^2}{\partial t'^2}$$

$$\nabla^2\phi = \nabla'^2\phi = \frac{1}{c^2}\cdot v^2\frac{\partial^2\phi}{\partial x'^2} - \frac{2v}{c}\frac{\partial^2\phi}{\partial x'\partial t'} + \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

b) $\phi = F[x-(c-v)t] + G[x+(c+v)t]$ soluzioa bada uhin-ekuazioa bete behar du:

$$\nabla^2\phi = \frac{\partial^2 F}{\partial x'^2} + \cancel{\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}}^{\,0} + \frac{\partial^2 G}{\partial x^2} + \cancel{\frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z'^2}}^{\,0} = F'' + G''$$

$*\ \frac{\partial F}{\partial x} = \frac{\partial F}{\partial(x-(c-v)t)}\cdot\frac{\partial(x-(c-v)t)}{\partial x} = F'$ , $\frac{\partial^2 F}{\partial x^2} = F''$ , $\frac{\partial G}{\partial x} = \frac{\partial G}{\partial(x+(c+v)t)}\frac{\partial(x+(c+v)t)}{\partial x} = G'$ , $\frac{\partial^2 G}{\partial x^2} = G''$

$$\frac{\partial^2 F}{\partial x^2} = F'' \quad , \quad \frac{\partial^2 F}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial(x-(c-v)t)} \frac{\partial(x-(c-v)t)}{\partial t}\right) = -\frac{\partial}{\partial x}\left(F' \cdot (c-v)\right) = -F''(c-v)$$

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial t}\right) = \frac{\partial}{\partial t}\left(-F'(c-v)\right) = (c-v)^2 F''$$

$$\frac{\partial^2 G}{\partial x^2} = G'' \quad , \quad \frac{\partial^2 G}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial(x+(c+v)t)} \cdot \frac{\partial(x+(c+v)t)}{\partial t}\right) = \frac{\partial}{\partial x}\left(G' (c+v)\right) = G''(c+v)$$

$$\frac{\partial^2 G}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial t}\right) = \frac{\partial}{\partial t}\left(G'(c+v)\right) = G''(c+v)^2$$

$$\Rightarrow \quad \nabla^2 \phi = G'' + F'' = \frac{v^2}{c^2} F'' + \frac{v^2}{c^2} G'' + \frac{2v}{c^2}(c-v)F'' - \frac{2v}{c^2}(c+v) G'' + \frac{(c+v)^2}{c^2} G'' +$$

$$\frac{(c-v)^2}{c^2} F'' = F''\left(\cancel{\frac{v^2}{c^2}} + \cancel{\frac{2v}{c}} - \cancel{\frac{2v^2}{c^2}} + 1 + \cancel{\frac{v^2}{c^2}} - \cancel{\frac{2v}{c}}\right) + G''\left(\cancel{\frac{v^2}{c^2}} - \cancel{\frac{2v}{c}} - \cancel{\frac{2v^2}{c^2}} + 1 + \cancel{\frac{v^2}{c^2}} + \cancel{\frac{2v}{c}}\right) =$$

$F'' + G''$ ✓ (Soluzioa da)

6.2.1

1. Lege fisikoak berdinak dira erreferentzia-sistema guztietan.

2. Argiaren abiadura (interakzioaren hedapena) berdina da erreferentzia-sistema guztietan.

S sistema eta S' sistema ($v$ abiaduraz lugitzen $x^+$)

Erlazioak $x$ eta $x'$-ren artean lineala izan behar direla onartzen dugu, erreferentzia sistema guztietan higidura uniformea izan dadin.

$x = ax' + bt' + c$ dela suposatuko dugu.

$x = x' = 0$ $\quad t' = t = 0$ dela hartuko dugu $\Rightarrow c = 0 \leftrightarrow x = ax' + bt'$

Beste edozein aldiunetan S-ren jatorria $x=0$ posizioan dago $\Rightarrow ax' + bt' = 0 \rightarrow$

$x'/t' = -b/a$ eta honixe da S' sisteman O-ren abiadura $\Rightarrow$

S-ren jatorria

$x'/t' = -v \leftrightarrow v = b/a$

Beraz, $x = a(x' + vt')$ (1). Galileoren transformazioetan alderantzizko transformazioa

$-v$ abiadura $v$-ren ordez jarriz lortzen zen, suposatuko dugu

gauzak orokortuko

hemen ere: $x' = a(x - vt)$ (2)

$x = x' = 0$ jatorriak $t = t' = 0$ aldiunean ignitzako fotoia

$c$ aldaezina denez edozein Sistematan, $x = ct$ eta $x' = ct'$ bete behar da $\Rightarrow$ (1) $\Rightarrow$ $ct = a(ct' + vt')$

(2) $\Rightarrow$ $ct' = a(ct - vt)$

$$t' = \frac{a}{c}(ct - vt) \Leftrightarrow ct = a\left(act - a\cancel{vt} + \cancel{vat} - \frac{av^2}{c}t\right) \Leftrightarrow$$

$$ct = a^2\left(ct - \frac{v^2}{c}t\right) \Leftrightarrow a^2 = \frac{c}{c - \frac{v^2}{c}} = \frac{1}{1 - \frac{v^2}{c^2}} \Leftrightarrow a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$$

$$b = av = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = v\gamma$$

Beraz,
$$\begin{cases} x = \gamma(x' + vt') \quad , \quad x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \quad ; \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right) \end{cases}$$

$$* \quad x = \gamma(x' + vt') = \gamma \cdot \gamma(x - vt) + \gamma vt' = \gamma^2 x - v\gamma^2 t + \gamma vt' \Rightarrow \gamma vt' = v\gamma^2 t + x(1 - \gamma^2) \Rightarrow$$

$\beta = v/c$

$$t' = \gamma t + x\left(\frac{1}{v\gamma} - \frac{\gamma}{v}\right) = \gamma t + \frac{x}{v\gamma}(1 - \gamma^2) = \gamma t + \frac{x}{v\gamma}\left(1 - \frac{1}{1 - \beta^2}\right) = \gamma t + \frac{x}{v\gamma}\left(\frac{\cancel{1} - \beta^2 - \cancel{1}}{1 - \beta^2}\right) =$$

$$\gamma t + \frac{x}{v\gamma}\left(\frac{-\beta^2}{1 - \beta^2}\right) = \gamma t - \frac{\beta^2}{\cancel{\gamma} v}x \cdot \gamma^{\cancel{2}} = \gamma t - \frac{v}{c^2}x\gamma = \gamma\left(t - \frac{v}{c^2}x\right)$$

6.3)

Bi gertakariren arteko denbora-tartea.

$$\begin{cases} x' = \gamma(x - vt) \; ; \quad x = \gamma(x' + vt') \\ t' = \gamma(t - \frac{v}{c^2}x) \; ; \quad t = \gamma(t' + \frac{v}{c^2}x') \end{cases}$$

S-n $\Rightarrow$ $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

S'-n $\Rightarrow$ $\Delta s'^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2$

$$\Delta x' = \gamma(\Delta x - v\Delta t) \; , \quad \Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x) \; , \quad \Delta y' = \Delta y \; , \quad \Delta z' = \Delta z$$

$$\Rightarrow \Delta s'^2 = \gamma^2(\Delta x - v\Delta t)^2 + \Delta y^2 + \Delta z^2 - c^2\gamma^2(\Delta t - \frac{v}{c^2}\Delta x)^2 = \gamma^2(\Delta x^2 + v^2\Delta t^2 - 2\Delta x v \Delta t) +$$

$$\Delta y^2 + \Delta z^2 - c^2\gamma^2(\Delta t^2 + \frac{v^2}{c^4}\Delta x^2 - \frac{2v}{c^2}\Delta x \Delta t) = \gamma^2\Delta x^2 + v^2\gamma^2\Delta t^2 - 2\cancel{\Delta x v}\gamma^2 \Delta t +$$

$$\Delta y^2 + \Delta z^2 - c^2\gamma^2\Delta t^2 - \gamma^2\frac{v^2}{c^2}\Delta x^2 + 2v\gamma^2\cancel{\Delta x}\Delta t = \Delta x^2\cancel{\gamma^2}(1 - \cancel{\frac{v^2}{c^2}}) + \Delta t^2(v^2\gamma^2 - c^2\gamma^2) +$$

$$\Delta y^2 + \Delta z^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2(1 - \cancel{v^2/c^2})\cancel{\gamma^2} = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2 = \Delta s^2$$

Erlatibitatean $\Rightarrow$ $t_2' - t_1' = t_2 - t_1$ mugo de $\Leftrightarrow$ $c = \infty$ EZINBEZKOA

6.4)

$u_x \Rightarrow$ S sisteman neurtutako x-ren norabideko abiadura

$u'_x \Rightarrow$ S' sisteman neurtutako x'-ren norabideko abiadura.

$$u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt}\cdot\frac{dt}{dt'} = \gamma(u_x - v)\cdot\frac{1}{\frac{dt'}{dt}} = \frac{\cancel{\gamma}(u_x - v)}{\cancel{\gamma}(1 - \frac{v}{c^2}u_x)} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$\Rightarrow$ $u'_y$ eta $u'_z$.

6.5)

Newtonen bigarren legea $\Rightarrow$ $\vec{F} = \frac{d\vec{p}}{dt}$ $\Rightarrow$ Indarraren transformazio erlatibista:

$$\vec{p} = m\vec{u} = m(u_x\hat{\imath} + u_y\hat{\jmath} + u_z\hat{k}) \rightarrow p = m(u_x^2 + u_y^2 + u_z^2)^{1/2} = m\dot{r}$$

Momentu tetrabektorea $p_\mu = (\vec{p}, \frac{iE}{c})$ ; $E = mc^2 + T = m\gamma c^2$ ; $\gamma = \frac{1}{\sqrt{1 - \frac{\dot{r}^2}{c^2}}}$

Indar tetrabektorea $\Rightarrow$ $F_\mu = \frac{dP_\mu}{dt} = \frac{dP_\mu}{dt} = m\left(\frac{d(\gamma\dot{\vec{r}})}{dt}, ic\frac{d\gamma}{dt}\right) \overset{*'}{=} m\gamma\left(\ddot{\vec{r}} + \dot{\vec{r}}\gamma^2\frac{\dot{\vec{r}}\cdot\ddot{\vec{r}}}{c^2}, \gamma^2 i\frac{\dot{\vec{r}}\cdot\ddot{\vec{r}}}{c}\right)$

$P_\mu = m u_\mu = m\gamma(\dot{\vec{r}}, ic)$

$*' \quad \frac{d(\gamma\dot{\vec{r}})}{dt} = \gamma\ddot{\vec{r}} + \dot{\vec{r}}\gamma^3\frac{\dot{\vec{r}}\cdot\ddot{\vec{r}}}{c^2}$

$\gamma = \frac{1}{\sqrt{1-\frac{\dot{r}^2}{c^2}}} \rightarrow \frac{d\gamma}{dt} = \gamma^3\frac{\dot{\vec{r}}\cdot\ddot{\vec{r}}}{c^2}$

$$P'_\mu = A P_\mu = \begin{pmatrix} \gamma' & 0 & 0 & i\beta\gamma' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma' & 0 & 0 & \gamma' \end{pmatrix}\begin{pmatrix} m\gamma\dot{x} \\ m\gamma\dot{y} \\ m\gamma\dot{z} \\ icm\gamma \end{pmatrix}$$

$\beta = v/c$

$\gamma = \left(\sqrt{1-\beta^2}\right)^{-1}$

$F'_\mu = A F_\mu$

Abiadurarekiko paraleloa den indarra: $\gamma^3 m\dot{\vec{r}}\cdot\frac{\dot{\vec{r}}\cdot\ddot{\vec{r}}}{c^2} = \vec{F}_\parallel \rightarrow F'_\parallel = F_\parallel$

Abiadurarekiko perpendikularra den indarra: $m\gamma\ddot{\vec{r}} = \vec{F}_\perp \rightarrow F'_\perp = \frac{1}{\gamma}F_\perp$

6.6.)

$E = mc^2 \Rightarrow$ seriearen garapena $\beta$-ren funtzioan $\dot{r}$ txikietarako $(\beta\to 0)$ $\left(\gamma = \frac{1}{\sqrt{1-\beta^2}}\right)$

- $E = mc^2 = \gamma m_0 c^2 = m_0c^2 + T = m_0c^2\cdot\frac{1}{\sqrt{1-\beta^2}} \overset{*}{=} m_0c^2\left(1 + \underbrace{\frac{\beta^2}{2} + \frac{3\beta^4}{8} + \frac{5\beta^6}{16} + \frac{35\beta^8}{128} + \dots}_{T}\right) \overset{\beta\to 0}{\simeq}$

$m_0c^2 + m_0c^2\frac{\beta^2}{2} = m_0c^2 + m_0\frac{\dot{r}^2}{2}$

$* \quad x\to 0; \frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \frac{35x^8}{128} + \dots$

- Pausagunelo partikula: $E_0 = m_0c^2$ ; $\vec{v} = \dot{\vec{r}}$ abiaduraz higitzen dena; $E = m_0c^2 + m_0\frac{\dot{r}^2}{2}$

$E - E_0 = m_0\frac{\dot{r}^2}{2} = T$

6.7.)

$r = 20\,Km$ , $\tau = 1'5\cdot 10^{-6}$ (batez-besteko bizi-denbora) , $c$

Muoiaren erreferentzia sisteman: $d' = c\tau = 450\,m$

Lurraren erreferentzia sisteman: $\Delta t = \gamma\tau \underset{c=0}{=} \tau$

$E_{min} = E_0 = m_\mu\cdot c^2 = 200\,m_e\cdot c^2 = 1'638\cdot 10^{-11}\,J$

6.8.)

$$\nabla_\mu = \frac{\partial}{\partial x_\mu} \qquad x_\mu = (x, y, z, ict)$$

$$\begin{cases} x \to \partial/\partial x & z \to \partial/\partial z \\ y \to \partial/\partial y & t \to -\frac{1}{c^2}\,\partial/\partial t \end{cases}$$

$$\nabla_\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{-ic}{c^2}\frac{\partial}{\partial t}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{ic}\frac{\partial}{\partial t}\right)$$

Tehrakelitee =) bere modulua aldaezina =) $\Box^2 = \nabla^2_\mu = \Box'^2 = \nabla'^2_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2} = \nabla'^2 - \frac{1}{c^2}\frac{\partial}{\partial t'^2}$

(aldaezinak maxwellen ekuazioak)

6.9.)

a) $\vec{B}\cdot\vec{E}$ eta $E^2 - c^2B^2$ magnitudeak aldaezinak direla frogatu.

- $\vec{B}'\cdot\vec{E}' = B'_xE'_x + B'_yE'_y + B'_zE'_z = B_xE_x + \gamma^2(B_y + \frac{v}{c^2}E_z)(E_y - vB_z) + \gamma^2(E_z + vB_y)(B_z - \frac{v}{c^2}E_y) =$

$B_xE_x + \gamma^2\left[B_yE_y - v\cancel{B_yB_z} + \frac{v}{c^2}\cancel{E_zE_y} - \frac{v^2}{c^2}E_zB_z + E_zB_z - \frac{v}{c^2}\cancel{E_zE_y} + \cancel{vB_y}B_z - \frac{v^2}{c^2}E_yB_y\right] =$

$B_xE_x + \cancel{\gamma^2}\left[B_yE_y\cancel{(1-\frac{v^2}{c^2})} + E_zB_z\cancel{(1-\frac{v^2}{c^2})}\right] = B_xE_x + B_yE_y + E_zB_z = \vec{B}\cdot\vec{E}$

- $E'^2 - c^2B'^2 = E_x'^2 + E_y'^2 + E_z'^2 - c^2(B_x'^2 + B_y'^2 + B_z'^2) = E_x^2 + \gamma^2(E_y - vB_z)^2 + \gamma^2(E_z + vB_y)^2 +$

$-c^2B_x^2 - c^2\gamma^2(B_y + \frac{v}{c^2}E_z)^2 - c^2\gamma^2(B_z - \frac{v}{c^2}E_y)^2 = E_x^2 + \gamma^2\left[E_y^2 + v^2B_z^2 - 2\cancel{E_yvB_z} + E_z^2 + v^2B_y^2 +\right.$

$\left.2vB_yE_z - c^2B_y^2 - \frac{v^2}{c^2}E_z^2 - 2\cancel{vE_zB_y} - c^2B_z^2 - \frac{v^2}{c^2}E_y^2 + 2\cancel{vE_y}B_z\right] - c^2B_x^2 = E_x^2 +$

$\gamma\left[E_y^2(1-\frac{v^2}{c^2}) + E_z^2(1-\frac{v^2}{c^2}) + B_z^2(v^2-c^2) + B_y^2(v^2-c^2)\right] - c^2B_x^2 =$

$E_x^2 - c^2B_x^2 + E_y^2 + E_z^2 + \gamma\left[c^2(\frac{v^2}{c^2}-1)B_z^2 + c^2(\frac{v^2}{c^2}-1)B_y^2\right] = E_x^2 + E_y^2 + E_z^2 +$

$-c^2\left[B_x^2 + B_y^2 + B_z^2\right] = E^2 - c^2B^2$

c) $E' = 0$? eta $B' = 0$? (Zein baldintzapetan) ?

$\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}' = 0 \Leftrightarrow \vec{E}\perp\vec{B}$ erreferentzia sistemaren batean.

$E'^2 - c^2B'^2 = E^2 - c^2B^2 = -c^2B'^2 \le 0 \Leftarrow B^2c^2 - E^2 \geq 0$ erreferentzia sistema horretan.

$E_{\parallel} \Rightarrow \vec{v}$-ren paraleloa den zenmua ; $E_{\perp} \rightarrow \vec{v}$-ren perpendikularra den zenmua $\vec{E} = \cancel{\vec{E}_{\parallel}}^{\to 0} + \vec{E}_{\perp}$

- $\vec{E}_{\parallel} = \vec{E}_{\parallel}' = 0$ ; $\vec{E}_{\perp}' = \gamma [\vec{E}_{\perp} + \vec{v} \times \vec{B}] = 0 \Leftrightarrow \vec{E}_{\perp} = -\vec{v} \times \vec{B} = \vec{B} \times \vec{v} = \vec{E}$ $\searrow \vec{E} \perp \vec{B}$

$\vec{E} \cdot \vec{B} = (\cancel{\vec{E}_{\parallel}} + \vec{E}_{\perp})(\vec{B}_{\parallel} + \vec{B}_{\perp}) = \vec{E}_{\perp} \cdot \vec{B}_{\perp} = 0$ $\quad \hookrightarrow \vec{v} = \dfrac{\vec{E} \times \vec{B}}{B^2}$

$$\vec{B}_{\parallel} = \vec{B}_{\parallel}' \quad ; \quad \vec{B}_{\perp}' = \gamma \left[\vec{B}_{\perp} - \frac{\vec{v} \times \vec{E}}{c^2}\right] = \gamma \left[\vec{B}_{\perp} - \frac{\vec{v} \times (\vec{B} \times \vec{v})}{c^2}\right] = \gamma \left[\vec{B}_{\perp} - \frac{1}{c^2}(\vec{B}(\vec{v} \cdot \vec{v}) - \vec{v}(\vec{B} \cdot \vec{v}))\right] =$$

$$\gamma \left[\vec{B}_{\perp} - \frac{1}{c^2}(\vec{B} \cdot v^2 - \vec{v}(B_{\parallel} v)\right] = \gamma \left[\vec{B}_{\perp} - \frac{1}{c^2}(\cancel{\vec{B}_{\parallel} v^2} + \vec{B}_{\perp} v^2 - \cancel{\vec{B}_{\parallel} v^2})\right] = \gamma \left[\vec{B}_{\perp} - \vec{B}_{\perp} \frac{v^2}{c^2}\right] =$$

$$\gamma \cdot \left(1 - \frac{v^2}{c^2}\right) \vec{B}_{\perp} = \frac{1}{\sqrt{(1 - v^2/c^2)}} \vec{B}_{\perp} (1 - v^2/c^2) = \sqrt{1 - v^2/c^2}\, \vec{B}_{\perp} = \frac{\vec{B}_{\perp}}{\gamma} \quad ; \quad B_{\perp}' = \frac{B_{\perp}}{\gamma}$$

$$\updownarrow$$

$$v = c\sqrt{1 - \frac{B_{\perp}'^2}{B_{\perp}^2}}$$

- $\vec{B}' = 0$ ; $\vec{E} \cdot \vec{B} = 0$ ; $\vec{B}_{\parallel} = \vec{B}_{\parallel}' = 0$ $\qquad E'^2 - c^2 B'^2 = E'^2 = E^2 - c^2 B^2 \geq 0$

$\vec{B}_{\perp}' = \gamma \left[\vec{B}_{\perp} - \dfrac{\vec{v} \times \vec{E}}{c^2}\right] = 0 \Leftrightarrow \vec{B}_{\perp} = \vec{B} = \dfrac{\vec{v} \times \vec{E}}{c^2}$ ; $\vec{v} = \dfrac{\vec{B} \times \vec{E}}{E^2} \cdot c^2$ $\quad \left( v = \dfrac{BE}{E^2} c^2 = \dfrac{B}{E} c^2 \right)$

$$\vec{E}_{\parallel} = \vec{E}_{\parallel}' \quad ; \quad \vec{E}_{\perp}' = \gamma [\vec{E}_{\perp} + \vec{v} \times \vec{B}] = \gamma \left[\vec{E}_{\perp} + \vec{v} \times \frac{(\vec{v} \times \vec{E})}{c^2}\right] = \gamma \left[\vec{E}_{\perp} + \frac{\vec{v}}{c^2}(\vec{v} \cdot \vec{E}) - \frac{\vec{E}}{c^2} v^2\right] =$$

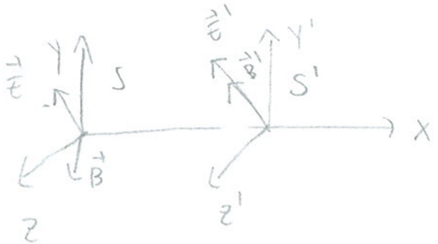
$$\gamma \left[\vec{E}_{\perp} + \frac{v}{c^2} \cdot v \vec{E}_{\parallel} - \frac{\vec{E}}{c^2} v^2\right] = \gamma \left[\vec{E}_{\perp} + \frac{v^2}{c^2}[\cancel{\vec{E}_{\parallel}} - \vec{E}_{\perp} - \cancel{\vec{E}_{\parallel}}]\right] = \gamma \left[\vec{E}_{\perp} - \frac{v^2}{c^2} \vec{E}_{\perp}\right] =$$

$$\gamma \vec{E}_{\perp} \left(1 - \frac{v^2}{c^2}\right) = \frac{\vec{E}_{\perp}}{\gamma} \Rightarrow v = c\sqrt{1 - \left(\frac{E_{\perp}'}{E_{\perp}}\right)^2}$$

b) $\vec{E} \cdot \vec{B} = \vec{E}' \cdot \vec{B}' \neq 0$ ; $E^2 - c^2 B^2 \neq 0$ (Erreferentzia sisteman batean $\vec{E} \parallel \vec{B}$)

$$\vec{E}' \cdot \vec{B}' = E_{\parallel}' B_{\parallel}' + \vec{E}_{\perp}' \cdot \vec{B}_{\perp}' = E_{\parallel} B_{\parallel} + \vec{E}_{\perp} \cdot \vec{B}_{\perp} = E' B' \left\{ \begin{array}{l} E_{\parallel}' = E_{\parallel} \\ B_{\parallel}' = B_{\parallel} \end{array} \right. \Leftrightarrow \vec{E}_{\perp}' \cdot \vec{B}_{\perp}' = \vec{E}_{\perp} \cdot \vec{B}_{\perp}$$

$$\vec{E}_{\perp}' = \gamma [\vec{E}_{\perp} + \vec{v} \times \vec{B}] \quad ; \quad \vec{B}_{\perp}' = \gamma \left[\vec{B}_{\perp} - \frac{\vec{v} \times \vec{E}}{c^2}\right] \Leftarrow \vec{E}_{\perp}' \cdot \vec{B}_{\perp}' = E' B' - E_{\parallel} B_{\parallel}$$



6.10.)

q Kargako partikula, V potentzial elektrostatikoa.

$$|q|V = \frac{1}{2}mv^2$$

Hasiera geldi zegoen $e^-$ bat $\Rightarrow V = 3\cdot 10^6$ V

M. Klasikoan $\Rightarrow$ $eV = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2eV}{m}} = 1'027\cdot 10^9$ (ezinezkoa, $v \geq c$)

M. erlatibista $\Rightarrow$ $eV + mc^2 = mc^2 + T \Leftrightarrow T = eV = m\gamma c^2 - mc^2 =$

$$mc^2(\gamma - 1) \Leftrightarrow \frac{eV + mc^2}{mc^2} = \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \left(\frac{eV+mc^2}{mc^2}\right)^2 = \frac{1}{1-\frac{v^2}{c^2}} \Rightarrow$$

$$1 - \frac{v^2}{c^2} = \left(\frac{mc^2}{eV+mc^2}\right)^2 \Leftrightarrow 1 - \left(\frac{mc^2}{eV+mc^2}\right)^2 = \frac{v^2}{c^2} \Rightarrow v = c\sqrt{1-\left(\frac{mc^2}{eV+mc^2}\right)^2} = 2'97\cdot 10^8 \text{ m/s}$$

6.11.)

a) Uhin lau monokromatikoa: $\vec{E}\cdot\vec{B} = 0 = E_xB_x + B_yE_y + E_zB_z$ , $|B| = \frac{|E|}{u}$ → hedapen abiadura.

- $\vec{E}'\cdot\vec{B}' = E_x'B_x' + E_y'B_y' + E_z'B_z' = \vec{E}\cdot\vec{B}$ (6.9 ariketan frogatuta) $\Rightarrow$ Beraz

$$\vec{E}\cdot\vec{B} = 0 = \vec{E}'\cdot\vec{B}' \qquad (\vec{E}', \vec{B}' \neq 0 \Rightarrow \vec{E}' \perp \vec{B}')$$

- $|B'|^2 = B_x'^2 + B_y'^2 + B_z'^2 = B_x^2 + \gamma^2\left(B_y^2 + \frac{v^2}{c^4}E_z^2 + \frac{2v}{c^2}B_yE_z + B_z^2 + \frac{v^2}{c^4}E_y^2 - 2B_z\frac{v}{c^2}E_y\right)$

$$|E'|^2 = E_x'^2 + E_y'^2 + E_z'^2 = E_x^2 + \gamma^2\left(E_y^2 + v^2B_z^2 - 2vE_yB_z + E_z^2 + v^2B_y^2 + 2vE_zB_y\right)$$

Hutsean hedatzen bada $\Rightarrow u = c \Leftrightarrow B = \frac{E}{c} \Leftrightarrow B^2 - \frac{E^2}{c^2}$

Frogatu dugu $B^2 - \frac{E^2}{c^2}$ aldaezina dela $\Leftrightarrow B^2 - \frac{E^2}{c^2} = B'^2 - \frac{E'^2}{c^2} = 0 \Leftrightarrow$

$$B' = \frac{E'}{c}$$ Bestela uste dut transformatu behar dela u

b) $\vec{K}\cdot\vec{r} - wt = kte = x_\mu \cdot k_\mu$ $\qquad (x_\mu = (\vec{r}, ict))$ erlatib.

↘ aldaezina ↗

$$k_\mu = \left(\vec{K}, \frac{-wt}{ict}\right) = \left(\vec{K}, i\frac{w}{c}\right)$$

tetrabektoreen biderkadura magnitude aldaezinak dira.

Edozein tetrabektoreren modulua konstante aldaezina da

6.12.)

$A_\mu$ (tetrabektorea), $A_\mu B_\mu$ aldaezina da $\Rightarrow$ $B_\mu$ tetrabektore bat da

$\vec{k}\cdot\vec{r}-\omega t = \omega e = x_\mu k_\mu$ $\quad (x_\mu = (\vec{r}, ict)) \Rightarrow k_\mu = (\vec{k}, i\frac{\omega}{c})$

a) $k'_\mu = A_{\mu\nu} k_\mu \Rightarrow$
$$\begin{pmatrix} k'_x \\ k'_y \\ k'_z \\ i\frac{\omega'}{c} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} k_x \\ k_y \\ k_z \\ i\frac{\omega}{c} \end{pmatrix} =$$

$$\begin{pmatrix} k_x\gamma - \beta\gamma\frac{\omega}{c} \\ k_y \\ k_z \\ -i\beta\gamma k_x + i\frac{\omega}{c}\gamma \end{pmatrix} \leftrightarrow \quad i\frac{\omega'}{c} = \frac{i\omega\gamma}{c} - i\beta\gamma k_x \leftrightarrow \omega' = \omega\gamma - \beta\cdot c\,\gamma k_x = \omega\gamma - v\gamma k_x =$$

$\rightarrow$ x - norabidealoa

$$\gamma(\omega - v k_x) = \frac{\omega - v k_x}{\sqrt{1-\beta^2}}$$

x norantzan hedatzen den uhin launa bada $\Rightarrow$ $k_x = k = \omega/c$ $\Rightarrow$

$$\omega' = \frac{\omega - v\,\omega/c}{\sqrt{1-\beta^2}} = \omega\frac{(1-\beta)}{\sqrt{1-\beta^2}} = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$

b) $\omega = 2\pi\nu \Rightarrow$ $\qquad \nu' = \nu\sqrt{\frac{1-\beta}{1+\beta}}$

$T = \frac{2\pi}{\omega}$, $T' = \frac{2\pi}{\omega'}$ $\Rightarrow$ $\qquad \frac{2\pi}{T'} = \frac{2\pi}{T}\sqrt{\frac{1-\beta}{1+\beta}} \leftrightarrow T' = T\sqrt{\frac{1+\beta}{1-\beta}}$

Urrundu $\Rightarrow$ $v>0 \Rightarrow \beta>0$ ; $\nu' = \nu\sqrt{\frac{1-\beta}{1+\beta}} < \nu$

Hurbildu $\Rightarrow$ $v<0 \Rightarrow \beta<0$ ; $\nu' = \nu\sqrt{\frac{1-\beta}{1+\beta}} = \nu\sqrt{\frac{1+|\beta|}{1-|\beta|}} > \nu$

6.13.)

$$T' = T\sqrt{\frac{1+\beta}{1-\beta}} = \frac{\lambda'}{c} = \frac{\lambda}{c}\sqrt{\frac{1+\beta}{1-\beta}} \Rightarrow \lambda' = \lambda\sqrt{\frac{1+\beta}{1-\beta}}$$

$\lambda' = 1'1\,\lambda \leftrightarrow$ $\qquad 1'1\,\cancel{\lambda} = \cancel{\lambda}\sqrt{\frac{1+\beta}{1-\beta}} \leftrightarrow (1'1)^2 = \frac{1+\beta}{1-\beta} \rightarrow 1'21(1-\beta) = 1+\beta \Rightarrow$

$1'21 - 1 = \beta(1+1'21) \Rightarrow$

$\frac{\lambda'-\lambda}{\lambda} = 0'1$

$v = 0'095c =$ $\Leftarrow$ $\beta = \frac{0'21}{2'21} = \frac{v}{c} = 0'095$

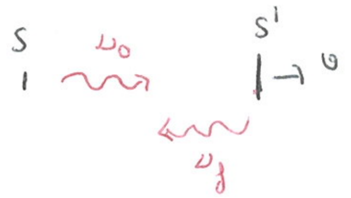
$2'85\cdot10^7$ m/s

6.14.)

$V_0$ frekuentzia ( S geldi̇rreko erreferentzia-sisteman igorritakoa) ; $\vec{v}$ abiaduraz mugitzen den ispilu baten kontra islatu ( $S'$ erreferentzia-sistema). $\nu_f$? (islatutakoa)

$$\nu' = \nu_0 \sqrt{\frac{1-\beta}{1+\beta}}$$ ispiluan jasotakoa $\Rightarrow$ islatu, $$\nu_f = \nu' \sqrt{\frac{1+v/c}{1-(v/c)}} = \nu' \sqrt{\frac{1-\beta}{1+\beta}} = \nu_0 \left(\frac{1-\beta}{1+\beta}\right)$$

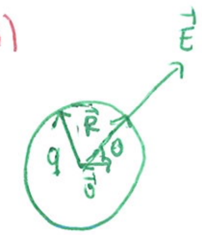
$\left(\beta = \frac{v}{c}\right)$



$- \ \nu_f > \nu_0 \iff \frac{1-\beta}{1+\beta} > 0 \Rightarrow 1-\beta > 1+\beta \Rightarrow \beta < 0$ ( hurbiltzen bada)

$- \ \nu_f < \nu_0 \iff \frac{1-\beta}{1+\beta} < 0 \Rightarrow 1-\beta < 1+\beta \Rightarrow \beta > 0$ (urruntzen bada)

6.15)



$v$ abiadura uniformez mugitzen den partikula

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0\gamma^2}\int_0^\pi \frac{dS}{R^2(1-\beta^2\sin^2\theta)^{3/2}} = \frac{q}{4\pi\varepsilon_0\gamma^2}\int_0^\pi \frac{2\pi \cancel{R^2}\sin\theta\, d\theta}{\cancel{R^2}(1-\beta^2\sin^2\theta)^{3/2}} =$$

$$\frac{q}{2\varepsilon_0\gamma^2}\int_0^\pi \frac{\sin\theta\, d\theta}{(\underbrace{1-\beta^2}_{1/\gamma^2}+\beta^2\cos^2\theta)^{3/2}} = \frac{q}{2\varepsilon_0\gamma^2}\int_0^\pi \frac{\sin\theta\, d\theta}{\left(\left(\frac{1}{\gamma}\right)^2+\beta^2\cos^2\theta\right)^{3/2}} \overset{*1}{=}$$

tauletan

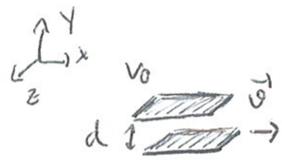
$$\frac{q}{2\varepsilon_0\gamma^2}\int_{-1}^{1} \frac{dx}{(\beta^2x^2+(1/\gamma)^2)^{3/2}} = \frac{q}{\varepsilon_0}$$

$*^1$ $x = \cos\theta$ , $dx = -\sin\theta\, d\theta$

$\theta = 0 \to x = 1$

$\theta = \pi \to x = -1$

6.16.)



Xafla paraleloko kondentsadorea $\Rightarrow \vec{v}$ abiaduraz mugitu

$Y' \in [0, d']$

$V_0' \Rightarrow$ potentzial diferentzia kondentsadorearen pausaguneko sisteman $\left( V_0 = \frac{d'Q}{\varepsilon_0 A'} \right)$ , $\phi_{Y'} = V_0 \frac{Y'}{d} = \frac{QY'}{\varepsilon_0 A'}$

$d \Rightarrow$ xaflen arteko distantzia kondentsadorearen pausaguneko sisteman.

Laborategiko sisteman: $d = d'$ mantentzen da baina azalera aldatzen da $\Rightarrow$

$A = A'/\gamma$ , beraz $$\phi_Y = \frac{QY}{\varepsilon_0 A} = \frac{QY'}{A'\varepsilon_0}\gamma = \phi_Y'\gamma = V_0\frac{Y}{d}\gamma \quad \left(\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

edo tetrabektorea $\Rightarrow$ matrizearekin $\phi = A_4 \cdot \frac{c}{i} \to A'_4 = \sum_{\mu} a_{4i} A_i$

Eremua:

Kondentsadorearen sisteman, $\vec{E}' = \frac{V_0}{d'}\hat{j}'$ , $\vec{E}$ (laborategian neurtutakoa)

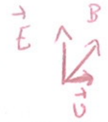
$E'_x = E_x = 0$ , $E_y = \gamma[E'_y + vB'_z] = \gamma E'_y$, $E_z = \gamma[E'_z - vB'_y] = 0$

$$\vec{E} = \gamma\vec{E}' = \gamma\frac{V_0}{d}\hat{j}$$

$B'_x = B_x = 0$ , $B_y = \gamma[B'_y + \frac{v}{c^2}E'_z] = 0$, $B_z = \gamma[B'_z - \frac{v}{c^2}E'_y] = \frac{\gamma v}{c^2}E'_y$ ;

$\vec{B}_z = \frac{\gamma}{c^2}\vec{v}\times\vec{E} = \frac{\gamma v E}{c^2}\hat{k} = \frac{\gamma v V_0}{dc^2}\hat{k}$ (S-n $\sigma$ karga dentsitatea ikusten da, $\vec{v}$ abiaduraz

↳ $\sigma = E'\cdot\varepsilon_0 = \frac{V_0}{d}\varepsilon_0$

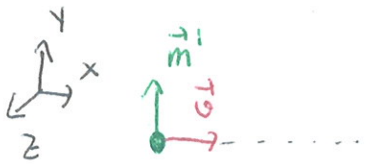



mugituz ⇒ korronte dentsitate bat ($K = \sigma\cdot v$) ⇒ eremu magnetikoa sortu)

↳ $B = \mu_0 K = \mu_0\sigma\cdot v = \mu_0\frac{V_0}{d}\varepsilon_0 v = \frac{V_0 v}{dc^2} = E'\frac{v}{c^2}$ (v<<c kasua)

6.17.)

Partikula puntuala ⇒ m momentu magnetikoa eta x ardatz positiboan mugitu.




Sortutako eremu elektrikoa → $E' = 0$

$\vec{r}' = x'\hat{i}' + y'\hat{j}' + z\hat{k}'$

S'-n

$$\vec{B}'(\vec{r}') = \frac{\mu_0}{4\pi r'^3}\left(\frac{3\vec{r}'(\vec{m}'\cdot\vec{r}')}{r'^2} - \vec{m}'\right) = \frac{\mu_0}{4\pi r'^3}\left(\frac{3\vec{r}'(m'\hat{j}\cdot\vec{r}')}{r'^2} - m'\hat{j}\right) =$$

$$\frac{\mu_0}{4\pi r'^3}\left(\frac{3\vec{r}'(m'y')}{r'^2} - m'\hat{j}'\right) = \frac{\mu_0 m'}{4\pi r'^3}\left(\frac{3(x'\hat{i}+y'\hat{j}+z'\hat{k})y}{x'^2+y'^2+z'^2} - \hat{j}'\right)$$


$$B'_x = \frac{\mu_0 m' y'}{4\pi r'^5}(3x') \quad , \quad B'_y = \frac{\mu_0 m'}{4\pi r'^3}\left(\frac{3y'^2}{r'^2} - 1\right)$$

$$B'_z = \frac{\mu_0 m' y'}{4\pi r'^5}(3z')$$

* edo $\hat{r}' = \hat{i}'$ hartu eta $\vec{m}'\cdot\hat{r}' = 0$

$$B_x = B'_x = \frac{\mu_0 m' 3x'y'}{4\pi r'^5} = \frac{3m'(x-vt)y\,\mu_0}{4\pi\left(\sqrt{(x-vt)^2+y^2+z^2}\right)^5}$$

$$B_y = \gamma B'_y = \gamma\frac{m'\mu_0}{4\pi r'^3}\left(\frac{3y'^2}{r'^2} - 1\right) = \gamma\frac{m'\left(\frac{3y^2}{(x-vt)^2+y^2+z^2} - 1\right)\mu_0}{4\pi\left((x-vt)^2+y^2+z^2\right)^{3/2}}$$

$$B_z = \gamma B_z' = \frac{\gamma m' y' \cdot 3z' \mu_0}{4\pi r'^5} = \frac{3\gamma m' y z \,\mu_0}{4\pi[\gamma^2(x-vt)^2+y^2+z^2]^{5/2}}$$

$\vec{r} = x\hat{i}$

$$E_x = E_x' = 0, \quad E_y = \gamma v B_z' = \gamma v \cdot \frac{3m' y z \,\mu_0}{4\pi[\gamma^2(x-vt)^2+y^2+z^2]^{5/2}}$$

$$\vec{E}_{dipolarra} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right] =$$

$$E_z = -\gamma v B_y' = -\gamma v \frac{m'\left(\frac{3y^2}{\gamma^2(x-vt)^2+y^2+z^2} - 1\right)\mu_0}{4\pi[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

$$\frac{-\vec{p}}{4\pi\varepsilon_0 x'^3} \to E_z = \frac{-p}{4\pi\varepsilon_0\gamma^3(x-vt)^3}$$

S-n momentu dipolar elektrikoa ikusten da, $p_z = -v\mu_0\gamma m'\varepsilon_0 = -\frac{v}{c^2}\gamma m' = -\frac{v}{c^2}\gamma \cdot m$

$\Rightarrow (x_0, 0, 0)$ puntuan:

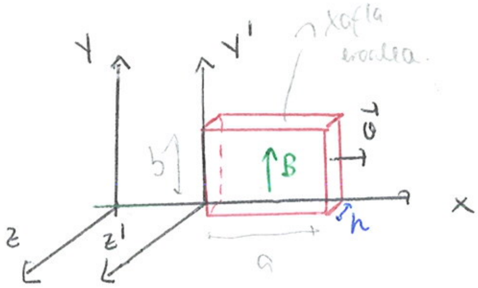
$\vec{p} = \left(\vec{m} \times \frac{\vec{v}}{c^2}\right)$

$$E_x = 0, \quad E_z = \frac{+\gamma v\, m'\mu_0}{4\pi(x_0-vt)^3\gamma^3}, \quad E_y = 0 \Rightarrow \vec{E} = \frac{m' v \mu_0}{4\pi(x_0-vt)^3\gamma^2}\hat{k}$$

$$B_x = 0, \quad B_y = \frac{-\gamma m' \mu_0}{4\pi(x_0-vt)^3\gamma^3}, \quad B_z = 0 \Rightarrow \vec{B} = \frac{-m'\mu_0}{4\pi(x_0-vt)^3\gamma^2}\hat{j}$$

momentu dipolar magnetikoak sortutako eremua x ardatzean.

6.18.)

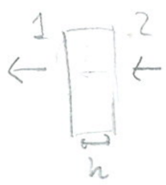



a) Induzitutako karga-dentsitatea?

xaflan: $B' = \gamma B \Rightarrow \vec{B}' = \gamma B\hat{j}$

$E'_x = E_x = 0, \quad E'_y = 0, \quad E_z' = \gamma v B_y \Rightarrow \vec{E}' = \gamma\vec{v}\times\vec{B} = \gamma v B\hat{k}$

$\vec{E}'$-k karga induzitu planoan:




$$\sigma_1' = \frac{E'}{\varepsilon_0} = \frac{\gamma v B}{\varepsilon_0}, \quad \sigma_2' = \frac{-E'}{\varepsilon_0} = \frac{-\gamma v B}{\varepsilon_0}$$

b) Laborategiko erreferentzia sisteman. $\sigma_1 = \sigma_1'\gamma, \quad \sigma_2 = \sigma_2'\gamma$

$$Q_1' = Q_1 = \sigma_1' \cdot a' \cdot b' = \sigma_1' \cancel{a' \cdot b} = \sigma_1 \cdot a \cdot b = \sigma_1 \cdot \frac{a'}{\gamma} b$$

6.19.)

eremu-elektromagnetiko-tentsorea

$$T_{\mu\nu} = F_{\mu\alpha} F_{\alpha\nu} + \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta}$$

energia-bultzada tentsorea

# ELEKTROMAGNETISMOA II azterketak:

## 2014-ko urtarrila:

3)

| | $T$ (K) | $\varepsilon_r$ |
|---|---|---|
| $CHCl_3$ (kloroformo) $\Rightarrow$ likidoa. | $T_1 = 293K$ | $\varepsilon_{r1} = 4'8$ |
| | $T_2 = 373K$ | $\varepsilon_{r2} = 3'7$ |

$p_0$?

Lehendabizi $N$ (molekula / V) kalkulatuko dugu: $N = \frac{1'48\, g}{cm^3} \cdot \frac{1\, mol}{119'35\, g} \cdot \frac{N_A \text{ molekula}}{1\, mol} =$

$$\frac{1'48 \cdot N_A}{119'35} \text{ molekula}/cm^3 = 7'31 \cdot 10^{21} \frac{\text{molekula}}{cm^3} \cdot \frac{1\, cm^3}{10^{-6}\, m^3} = 7'31 \cdot 10^{27} \text{ molekula}/m^3$$

Bestalde, $\alpha = \alpha_{ind.} + \alpha_{orient} = \alpha_{ind} + \frac{p_0^2}{3k_B T}$, eta Clausius-Mossotti-ren ekuazioa erabiliz:

$$\alpha = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = \alpha_{ind} + \frac{p_0^2}{3k_B T}$$

Beraz, $\alpha(T_1) - \alpha(T_2) = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_{r1}-1}{\varepsilon_{r1}+2} - \frac{\varepsilon_{r2}-1}{\varepsilon_{r2}+2}\right) = \frac{p_0^2}{3k_B}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \Rightarrow$

$$\boxed{p_0 = \sqrt{\frac{9k_B\varepsilon_0\left(\frac{\varepsilon_{r1}-1}{\varepsilon_{r1}+2} - \frac{\varepsilon_{r2}-1}{\varepsilon_{r2}+2}\right)}{N\left(\frac{1}{T_1} - \frac{1}{T_2}\right)}} = 4'18 \cdot 10^{-30} \quad C \cdot m}$$

Teorikoa, onartutakoa: $p_0\,(teo) = 3'8 \cdot 10^{-30}\ C\cdot m \Rightarrow E_r = \frac{p_0(neurtu) - p_0(teo)}{p_0(teo)} = 0'1$

## Galderak:

4.c)

Uhin-eroako antena $\Rightarrow \langle P_r \rangle = 73'1 \frac{I_0^2}{2}$ (denboran batezbestekoa)

$r = 10\, km$, $\theta = 90° = \pi/2 \Rightarrow E = 1\, V/m$ erradiazio eremuan gaudela suposatuz.

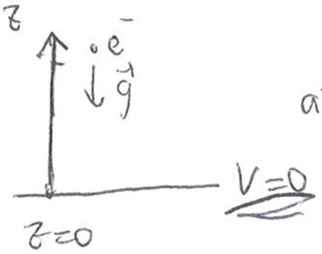
$$\vec{E} = \vec{E}_\theta = \frac{1}{4\pi\varepsilon_0}\frac{I_0}{c}\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\frac{e^{i(kr-\omega t)}}{r}\hat{u}_\theta \Rightarrow \vec{E}(\pi/2, r=10km) = \frac{I_0}{4\pi\varepsilon_0 \cdot c} \cdot \frac{2e^{i(kr-\omega t)}}{r}$$

$\frac{I_0}{4\pi\varepsilon_0}\cdot\frac{2}{cr} = E = 1\,V/m \Rightarrow \boxed{I_0 = \frac{4\pi\varepsilon_0 E c r}{2} = 2'11\cdot10^{-6}\,A} \Rightarrow \langle P_r\rangle = 73'1\,\frac{I_0^2}{2} = \boxed{1'63\cdot10^{-10}\,W}$

2012 / X / 22 ⇒ 2. partziala

1.)

$e^-$ bat ⇒ grabitatearen eraginez erori ⇒ partikula kargatu bat azeleratuko da ⇒ erradiazioa igorriko du.



a) Energiaren balantzea ⇒ (energiaren kontserbazioa)

$$\cancel{\frac{d}{dt}E_{tot}}\ \underbrace{T+V}_{e^-} + E_{rr} = kte \Rightarrow$$

energia grabitatorioa galtzen ⇒ abiadura handituz eta erradiazioa sortuz.

$$\frac{d}{dt}(T+V) + P_{rr} = 0 \Rightarrow$$

↗ orain $\ddot z = kte\ (g)$

$\frac{d}{dt}\left(\frac12 m\dot z^2 + mgz\right) + P_{rr} = 0$, $P_{rr}$ ⇒ Larmorren formula $P_{rr} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\ddot z^2$

$$\boxed{\dot z m\ddot z + mg\dot z + \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\ddot z^2 = 0}$$ ⇒ horrendako ? (adu. i z(t))

b) $P_{rr} = +\frac{dW_{rr}}{dt} = -\frac{dW}{dt}$ → Energia galera ⇒ $\Delta W_{rr} = P_{rr}\cdot\Delta t$

$\frac{dV}{dt} = mg\dot z \Rightarrow \Delta V = mg\Delta z.$ ⇒ $\frac{-\Delta W_{rr}}{\Delta V} = -\frac{\Delta W}{\Delta V} = \frac{P_{rr}\Delta t}{mg\Delta z}$

$\Delta z = \cancel{v_0 t} + \frac12 a t^2 \Rightarrow \Delta t = \sqrt{\frac{2\Delta z}{g}}$ → hurbilketa ona

Dn Elektroia $g$ azelerazio konstantez azeleratzen dela suposatuz: $P_{rr} = \frac{dW_{rr}}{dt} \Rightarrow$

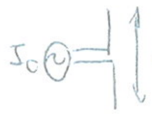
$-W = +W_{rr} = \frac{q^2 g^2}{6\pi\varepsilon_0 c^3}\Delta t + \cancel{K}^{0}$ ⇒ energia galera ⇒ $\Delta W = \frac{-q^2 g^2}{6\pi\varepsilon_0 c^3}\Delta t$

$\Delta V = +mg\Delta z \Rightarrow \frac{\Delta W}{\Delta V} = -\frac{\Delta W_{rr}}{mg\Delta z} = \frac{-q^2 g^2}{6\pi\varepsilon_0 c^3}\frac{\Delta t}{mg\Delta z}$

$\Delta z < 0$ (erori)

2.)

$l = 30m$ (AM mota) , $\nu = 5\ MHz$ , $I_0 = 20\ A$




a) $\lambda = \frac{c}{\nu} = \frac{3\cdot 10^8 m/s}{5\ s^{-1}\cdot 10^6} = 0'6\cdot 10^2 m = 60m \Rightarrow \frac{\lambda}{l} = \frac{60m}{30m} = 2 \leftrightarrow \frac{l}{\lambda} = \frac{1}{2} \rightarrow l = \lambda/2$

Uhin-erdiko antena.

b) $\langle P_r \rangle$? $r = 2\ Km >> \lambda \Rightarrow$ erradiazioko eremuan:

$\langle P_r \rangle = 73'1 \frac{I_0^2}{2} = 1'462\cdot 10^4\ W$ ; $E_{0\,max} = \frac{1}{4\pi\varepsilon_0}\frac{I_0}{c}\cdot\frac{2\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta\ r} \Rightarrow$ maxima

$\theta = \pm\pi/2 \Rightarrow |E_{0\,max}| = \frac{I_0}{4\pi\varepsilon_0}\frac{2}{cr} \underset{r=2km}{=} \frac{I_0}{2\pi\varepsilon_0 c r} = \frac{20A}{2\pi\cdot 8'85\cdot 10^{-12}\cdot 3\cdot 10^8\cdot 2\cdot 10^3} = 0'599\ V/m$

3.) $\omega = 100\ MHz = 10^8\ Hz.$

$l = 2mm$ (dipolo elektrikoa) $\Rightarrow \vec{p} = q\vec{l} \Rightarrow \dot{p} = \dot{q}\vec{l} = I\vec{l} = I_0 e^{i\omega t}\vec{l}$

$a = 2mm$ erradioko (dipolo magnetikoa) $\Rightarrow \vec{m} = I\cdot\pi a^2 \rightarrow \dot{\vec{m}} = I\cdot i\omega\pi a^2$

Dipolo elek. $\Rightarrow \langle P_r \rangle_{d.e} = \frac{1}{4\pi\varepsilon_0}\frac{\omega^4}{c^3}\cdot\frac{1}{3}p_0^2 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{3}\frac{I_0^2 l^2\omega^2}{c^3}$

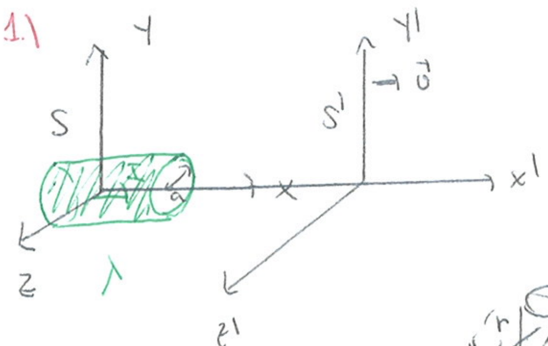
$p_0 = \frac{I_0 l}{\omega}$

Dipolo mag $\Rightarrow \langle P_r \rangle_{d.m} = \frac{\mu_0}{4\pi}\cdot\frac{1}{3}\frac{m_0^2\omega^4}{c^3} = \frac{\mu_0}{4\pi}\cdot\frac{1}{3}\frac{I_0\pi^2 a^4\omega^4}{c^3} = \ldots$

$m_0 = I_0\pi a^2$

2012 / XII / 17

1.)




$\lambda \Rightarrow$ lineako dentsitate karga , I uniformeki banatuko I korronte intentsitatea du.

Zilindroa pausagunean dago S sisteman:

S sistemaren neurtuko eremuak:


Gauss $\Rightarrow E\cdot 2\pi r L = \frac{\lambda L}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$ $(r>a)$

Ampère $\Rightarrow B 2\pi r = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{u}_\varphi$ $(r>a)$

$$r=\sqrt{x^2+y^2+z^2}=\sqrt{\gamma^2(x'+vt')^2+y'^2+z'^2}$$ (B i [illegible] $\vec{v}$-re perpendikuláris)

$$S'\text{-n} \Rightarrow \vec{E}_{\parallel}'=\vec{E}_{\parallel}=0\ ,\ \vec{E}_{\perp}'=\gamma[\vec{E}_{\perp}+\vec{v}\times\vec{B}]=\gamma[\vec{E}-vB\hat{r}]$$

$$E_r'=\gamma[E_r-vB_\varphi]=\gamma\left[\frac{\lambda}{2\pi\varepsilon_0 r}-v\cdot\frac{\mu_0 I}{2\pi r}\right]=\frac{\lambda\gamma}{2\pi\varepsilon_0 r}-\frac{v\mu_0 I\gamma}{2\pi r}=$$

$$\frac{\lambda\gamma}{2\pi\varepsilon_0\sqrt{\gamma^2(x'+vt')^2+y'^2+z'^2}}-\frac{v\mu_0 I\gamma}{2\pi\sqrt{\gamma^2(x'+vt')^2+y'^2+z'^2}}$$

$$\vec{B}_{\parallel}'=\vec{B}_{\parallel}=0\ ;\quad \vec{B}_{\perp}'=\gamma[\vec{B}_{\perp}-\frac{\vec{v}}{c^2}\times\vec{E}]=\gamma[\vec{B}_{\perp}-\frac{v}{c^2}E\,\hat{u}_\varphi]\Rightarrow$$

$$B_\varphi'=\gamma[B_\varphi-\frac{v}{c^2}E_r]=\gamma\left[\frac{\mu_0 I}{2\pi r}-\frac{v}{c^2}\frac{\lambda}{2\pi\varepsilon_r}\right]$$

a) $v=\dfrac{\lambda}{\varepsilon_0\mu_0 I}\Rightarrow$ $$E_r'=\frac{\lambda\gamma}{2\pi\varepsilon_0 r}-\frac{\lambda}{\varepsilon_0\cancel{\mu_0 I}}\cdot\frac{\cancel{\mu_0 I}\gamma}{2\pi r}=0$$

$$B_\varphi'=\gamma\left[\frac{\mu_0 I}{2\pi r}-\frac{\lambda}{\cancel{\varepsilon_0}\mu_0 I\cancel{c^2}}\frac{\lambda}{2\pi\varepsilon_0 r}\right]=\frac{\mu_0 I}{2\pi r}\gamma\left[1-\frac{\lambda^2c^2}{I^2}\right]=\frac{B_\varphi}{\gamma}$$

b) $v=\dfrac{I}{\lambda}\Rightarrow$ $$E_r'=\frac{\lambda\gamma}{2\pi\varepsilon_0 r}-\frac{I}{\lambda}\cdot\frac{\mu_0 I\gamma}{2\pi r}=\frac{\lambda\gamma}{2\pi\varepsilon_0 r}\left[1-\frac{I^2}{\lambda c^2}\right]=\frac{E_r}{\gamma}$$

$$\mu_0=\frac{1}{c^2\varepsilon_0}$$

$$B_\varphi'=\gamma\left[\frac{\mu_0 I}{2\pi r}-\frac{I^{\cancel{2}}}{\cancel{\lambda^2}c^2}\frac{\cancel{\lambda}}{2\pi\varepsilon_0 r}\right]=0$$

2.)

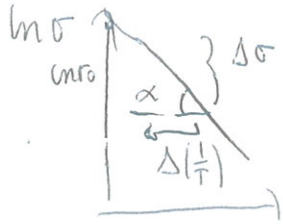
| $T$ (K) | $\sigma$ $(\Omega\cdot m)^{-1}$ |
|---|---|
| $T_1=293$ | $\sigma_1=250$ |
| $T_2=373$ | $\sigma_2=1000$ |

$$\sigma=nq(\mu_e+\mu_h)=n_0e^{-E_g/2k_BT}q(\mu_e+\mu_h)=$$
$$Ae^{-E_g/2k_BT}\Leftrightarrow$$

$$\frac{-E_g}{2k_BT}=\ln\left(\frac{\sigma}{A}\right)\Leftrightarrow E_g=2k_BT\ln\left(\frac{A}{\sigma}\right)\Leftrightarrow \ln\sigma=\ln\sigma_0-E_g/2k_BT$$

$$\frac{\sigma_1}{\sigma_2}=\frac{e^{-E_g/2k_BT_1}}{e^{-E_g/2k_BT_2}}=e^{-\frac{E_g}{2k_B}\left(\frac{1}{T_1}-\frac{1}{T_2}\right)}\Leftrightarrow \ln\left(\frac{\sigma_1}{\sigma_2}\right)=\frac{-E_g}{2k_B}\left(\frac{1}{T_1}-\frac{1}{T_2}\right)$$

$$\frac{\ln\left(\frac{\sigma_2}{\sigma_1}\right)\cdot 2k_B}{\left(\frac{1}{T_1}-\frac{1}{T_2}\right)} = \boxed{E_g = 5'22\cdot 10^{-20}\,\cancel{J}\cdot\frac{1\,eV}{1'6\cdot 10^{-19}\,\cancel{J}} = 0'3266\ eV}$$



$$tg\,\alpha = \frac{+E_g}{2k_B} = \frac{-\Delta\sigma}{\Delta\left(\frac{1}{T}\right)} = \frac{-\ln\sigma_1 - \ln\sigma_2}{\frac{1}{T_1}-\frac{1}{T_2}} = \frac{\ln\left(\frac{\sigma_2}{\sigma_1}\right)}{\frac{1}{T_1}-\frac{1}{T_2}}$$

3.)

| | T (K) | $\varepsilon_r$ |
|---|---|---|
| $NH_3 \Rightarrow$ | $T_1 = 273K$ | $1'00834$ |
| | $T_2 = 373K$ | $1'00487$ |

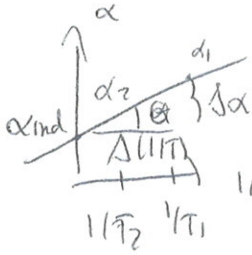
$M_m(NH_3) = 17\ g/mol$

$d(NH_3) = 0'86\ g/L = 0'86\cdot 10^{+3}\ g/m^3$

$$N = 0'86\cdot 10^{+3}\frac{\cancel{g}}{m^3}\cdot\frac{1\,\cancel{mol}}{17\,\cancel{g}}\cdot\frac{N_A\ \text{moléculas}}{mol} = 3'046\cdot 10^{25}\ \text{moléculas}/m^3$$

$$\alpha = \alpha_{ind} + \alpha_{ent} = \alpha_{ind} + \frac{p_0^2}{3k_BT} = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)$$

↳ Claussius - Mossotti

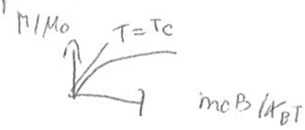


$$tg\,\theta = \frac{p_0^2}{3k_B} = \frac{\Delta\alpha}{\Delta\left(\frac{1}{T}\right)} = \frac{\alpha(T_1)-\alpha(T_2)}{\frac{1}{T_1}-\frac{1}{T_2}} = \frac{\frac{3\varepsilon_0}{N}\left[\frac{\varepsilon_{r1}-1}{\varepsilon_{r1}+2} - \frac{\varepsilon_{r2}-1}{\varepsilon_{r2}+2}\right]}{\frac{1}{T_1}-\frac{1}{T_2}} =$$

$\cancel{8'316\cdot 10^{-35}}$ $\quad 1'022\cdot 10^{-36}$ $\Rightarrow \boxed{p_0 = 6'55\cdot 10^{-30}\ C\cdot m}$

4.)

Imanes - ferromagnético $\Rightarrow$ $\chi(T)$? $\quad T >> T_C$ (Curie paramagnetismo):




$$H_0 = H - H_m = H - \gamma M$$

$$M = M_0\,\mathcal{L}(Bm_0/k_BT) = M_0\,\mathcal{L}(\mu_0 H m_0/k_BT) = M_0\,\mathcal{L}\left(\frac{\mu_0 m_0}{k_BT}(H_0 + H_m)\right) =$$

$$M_0\,\mathcal{L}\left(\frac{\mu_0 m_0}{k_BT}(H_0 + \gamma M)\right) = M_0\,\mathcal{L}\Big(\underbrace{\frac{\mu_0 m_0 H_0}{k_BT} + \frac{\gamma M \mu_0 m_0}{k_BT}}_{x}\Big)$$

$T >> T_C \Rightarrow \mathcal{L}(x) = \frac{x}{3} \Rightarrow M = M_0\frac{x}{3} = \frac{M_0}{3}\frac{\mu_0 m_0 H_0}{k_BT} + \frac{M_0}{3}\gamma\frac{M\mu_0 m_0}{k_BT}$

$x << 1$,

2013/11/17.

2.)

$E_0 = 100$ keV ⇒ $e^-$ da ⇒ azeleragailu zirkularrean sartu. ⇒ $R = 5$m

$B = 1$ T ⇒ $\vec{B} \perp \vec{v}$  $(\vec{F} = -e\vec{v} \times \vec{B})$  Bira batean erradiatutako P eta W?

(Bira batean ⇒ $T = \Delta t$)


$evB = m\frac{v^2}{R} \Rightarrow \cancel{v = \frac{ReB}{m}}$

$R = \frac{mv}{eB}$ ⇒

$E_0 = \frac{1}{2}mv_0^2 \Rightarrow v_0 = \sqrt{\frac{2E_0}{m}} = 1'87\cdot10^8$ m/s

↓ azeleratu.

~~$v = 8'79\cdot10^{11}$ m/s~~

azelerazio batean

$R_0 = \frac{mv}{eB} = 1'066\cdot10^{-3}$ m $= 1'066$ mm $< R = 5$m ⇒ $a_n = \frac{v_0^2}{R_0} = 3'278\cdot10^{19}$ m/s²

Larmor

$P_r = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}\frac{e^2}{c^3}a^2 = 6'11\cdot10^{-15}$ W ⇒ $\Delta t = \frac{2\pi}{\omega} = \frac{2\pi R}{v} = 3'58\cdot10^{-11}$ s

$E_{irr} = P_r\cdot\Delta t = 2'19\cdot10^{-25}$ J

Galderak:

4.) c) $\sigma = 37'7\cdot10^6\ (\Omega m)^{-1}$ ⇒ Al atomo bakoitzean. $\mu = 13\cdot10^{-4}\ m^2/(V\cdot s)$

$\sigma = Nq\mu \Rightarrow \cancel{N = \frac{\sigma}{q\mu} = \frac{\sigma}{n_e e\mu}}$

$\sigma = N\cdot n_e\cdot e\mu$ ⇒ $n_e = \frac{\sigma}{Ne\mu} = 3\ e^-$ $(Al^{3+})$

↳ $e^-$ kopurua

$N = \frac{2'7\cdot10^6\ g}{m^3}\cdot\frac{1\ mol}{27\ g}\cdot\frac{N_A\ molekula}{mol} = 6'022\cdot10^{28}$ molekula/m³

# 3. UHIN ELEKTROMAGNETIKOAK INGURUNE MUGATUETAN

16-10-31

## UHIN ELEKTROMAGNETIKOEN ISLAPENA eta TRANSMISIOA

3.1.)

Berezko inpedantzia → $\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}$ ; $\eta_i = \sqrt{\frac{\mu_i}{\varepsilon_i}} \underset{*}{=} \frac{\sqrt{\mu_i}}{n_i / c\sqrt{\mu_i}} = \frac{\mu_i c}{n_i} \Longrightarrow$

* $v_i = \frac{c}{n_i} = \frac{1}{\sqrt{\varepsilon_i \mu_i}} \rightarrow \sqrt{\varepsilon_i} = \frac{n_i}{c\sqrt{\mu_i}}$

ingurunean uhinaren abiadura

∘ $n_i = \frac{\mu_i c}{n_i} = \frac{\mu_0 \cdot \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}{\eta_i} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_0}}}{\eta_i} = \frac{\eta_0}{\eta_i}$

$\mu_i = \mu_0$ (ingurune ez-magnetikoak)

∘ Fresnell-en ekuazioak:

* $\frac{E_{rn}}{E_{in}} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} = \frac{\eta_0/\eta_1 \cos\theta_i - \eta_0/\eta_2 \cos\theta_t}{\eta_0/\eta_1 \cos\theta_i + \eta_0/\eta_2 \cos\theta_t} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{\cos\theta_i - \frac{\eta_1}{\eta_2}\cos\theta_t}{\cos\theta_i + \frac{\eta_1}{\eta_2}\cos\theta_t}$

* $\frac{E_{tn}}{E_{in}} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} = \frac{2\eta_0/\eta_1 \cos\theta_i}{\frac{\eta_0}{\eta_1}\cos\theta_i + \frac{\eta_0}{\eta_2}\cos\theta_t} = \frac{\frac{2}{\eta_1}\cos\theta_i}{\frac{\cos\theta_i}{\eta_1} + \frac{\cos\theta_t}{\eta_2}} = \frac{2\cos\theta_i}{\cos\theta_i + \frac{\eta_1}{\eta_2}\cos\theta_t}$

* $\frac{E_{rp}}{E_{ip}} = \frac{-n_2 \cos\theta_i + n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} = \frac{-\eta_0/\eta_2 \cos\theta_i + \eta_0/\eta_1 \cos\theta_t}{\eta_0/\eta_2 \cos\theta_i + \eta_0/\eta_1 \cos\theta_t} = \frac{\eta_2/\eta_1 \cos\theta_t - \cos\theta_i}{\cos\theta_i + \eta_2/\eta_1 \cos\theta_t}$

* $\frac{E_{tp}}{E_{ip}} = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} = \frac{2\eta_0/\eta_1 \cos\theta_i}{\eta_0/\eta_2 \cos\theta_i + \eta_0/\eta_1 \cos\theta_t} = \frac{2\cos\theta_i}{\eta_1/\eta_2 \cos\theta_i + \cos\theta_t}$

* $\frac{B_{rp}}{B_{ip}} = \frac{E_{rn}}{E_{in}} = \frac{\cos\theta_i - \eta_1/\eta_2 \cos\theta_t}{\cos\theta_i + \eta_1/\eta_2 \cos\theta_t}$

$$\ast\ \frac{B_{tp}}{B_{ip}} = \frac{n_2 E_{tn}}{n_1 E_{in}} = \frac{\eta_0/\eta_2}{\eta_0/\eta_1}\frac{E_{tn}}{E_{in}} = \frac{\eta_1}{\eta_2}\frac{E_{tn}}{E_{in}} = \frac{2\eta_1\cos\theta_i}{\eta_2(\cos\theta_i + \frac{\eta_1}{\eta_2}\cos\theta_t)} = \frac{2\eta_1\cos\theta_i}{\eta_2\cos\theta_i+\eta_1\cos\theta_t} =$$

$$\frac{2\cos\theta_i}{\cos\theta_t+\frac{\eta_2}{\eta_1}\cos\theta_i}$$

$$\ast\ \frac{B_{rn}}{B_{in}} = \frac{E_{rp}}{E_{ip}} = \frac{\eta_2/\eta_1\cos\theta_t - \cos\theta_i}{\eta_2/\eta_1\cos\theta_t + \cos\theta_i}$$

$$\ast\ \frac{B_{tn}}{B_{in}} = \frac{n_2 E_{tp}}{n_1 E_{ip}} = \frac{\eta_1}{\eta_2}\frac{E_{tp}}{E_{ip}} = \frac{\eta_1}{\eta_2}\frac{2\cos\theta_i}{\cos\theta_t+\frac{\eta_1}{\eta_2}\cos\theta_i} = \frac{2\cos\theta_i}{\cos\theta_i+\frac{\eta_2}{\eta_1}\cos\theta_t}$$

3.2)

Eraso normala → $\theta_i = \theta_r = \theta_t = 0$

$$\ast\ \frac{E_{rn}}{E_{in}} = \frac{n_1-n_2}{n_1+n_2} \qquad \ast\ \frac{E_{tn}}{E_{in}} = \frac{2n_1}{n_1+n_2} \qquad \ast\ \frac{B_{rp}}{B_{ip}} = \frac{E_{rn}}{E_{in}} = \frac{n_1-n_2}{n_1+n_2}$$

$$\ast\ \frac{E_{rp}}{E_{ip}} = \frac{n_1-n_2}{n_2+n_1} \qquad \ast\ \frac{B_{rn}}{B_{in}} = \frac{E_{rp}}{E_{ip}} = \frac{n_1-n_2}{n_1+n_2} \qquad \ast\ \frac{E_{tp}}{E_{ip}} = \frac{2n_1}{n_1+n_2}$$

$$\ast\ \frac{B_{tp}}{B_{ip}} = \frac{n_2}{n_1}\frac{E_{tn}}{E_{in}} = \frac{2n_1n_2}{n_1^2+n_2n_1} = \frac{2}{\frac{n_1}{n_2}+1}$$

$$\ast\ \frac{B_{tn}}{B_{in}} = \frac{n_2}{n_1}\frac{E_{tp}}{E_{ip}} = \frac{2n_1n_2}{n_1(n_1+n_2)} = \frac{2}{\frac{n_1}{n_2}+1}$$

3.3)

$$R = \frac{I_r}{I_i}\ ;\ T = \frac{I_t}{I_i} \Rightarrow R_n = \frac{I_{rn}}{I_{in}}\ ,\ R_p = \frac{I_{rp}}{I_{ip}}\ ,\ T_n = \frac{I_{tn}}{I_{in}}\ ,\ T_p = \frac{I_{tp}}{I_{ip}}$$

- $R_n = \frac{I_{rn}}{I_{in}} = \frac{E_{rn}^2}{E_{in}^2} = \frac{\sin^2(\theta_i-\theta_t)}{\sin^2(\theta_i+\theta_t)}$
- $R_p = \frac{I_{rp}}{I_{ip}} = \frac{E_{rp}^2}{E_{ip}^2} = \frac{\tan^2(\theta_i-\theta_t)}{\tan^2(\theta_i+\theta_t)}$
- $T_n = \frac{I_{tn}}{I_{in}} = \frac{n_2}{n_1}\cancel{\frac{E_{tn}^2}{E_{in}^2}}\ \frac{\sin 2\theta_i \sin 2\theta_t}{\sin^2(\theta_i+\theta_t)}$
- $T_p = \frac{I_{tp}}{I_{ip}} = \frac{n_2}{n_1}\cancel{\frac{E_{tp}^2}{E_{ip}^2}}\ \frac{\sin 2\theta_i \sin 2\theta_t}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)}$

$$\Rightarrow \quad R_n + T_n = \frac{\sin^2(\theta_i-\theta_t)}{\sin^2(\theta_i+\theta_t)} + \frac{\sin 2\theta_i \sin 2\theta_t}{\sin^2(\theta_i+\theta_t)} = \frac{\sin^2(\theta_i-\theta_t)+\sin 2\theta_i \sin 2\theta_t}{\sin^2(\theta_i+\theta_t)} \overset{*}{=} \frac{\frac{1}{2}(1-\cos(2\theta_i+2\theta_t))}{\sin^2(\theta_i+\theta_t)} =$$

$$\frac{\sin^2(\theta_i+\theta_t)}{\sin^2(\theta_i+\theta_t)} = 1$$

$* \ \sin 2\theta_i \sin 2\theta_t = \frac{1}{2}\cos(2\theta_i-2\theta_t) - \frac{1}{2}\cos(2\theta_i+2\theta_t)$

$\sin^2(\theta_i-\theta_t) = \frac{1}{2} - \frac{1}{2}\cos(2(\theta_i-\theta_t))$ ; $\cos^2(\theta_i+\theta_t) = \frac{1}{2}(1+\cos(2\theta_i+2\theta_t))$

$$\Rightarrow \quad R_p + T_p = \frac{\tan^2(\theta_i-\theta_t)}{\tan^2(\theta_i+\theta_t)} + \frac{\sin 2\theta_i \sin 2\theta_t}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)} = \frac{\sin^2(\theta_i-\theta_t)\cos^2(\theta_i+\theta_t)}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)} + \frac{\sin 2\theta_i \sin 2\theta_t}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)} \overset{*}{=}$$

$$\frac{\frac{1}{2}\cdot\frac{1}{2}(1-\cos(2\theta_i-2\theta_t))(1+\cos(2\theta_i+2\theta_t)) + \frac{1}{2}(\cos(2\theta_i-2\theta_t)-\cos(2\theta_i+2\theta_t))}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)} =$$

$$\frac{1}{2}\ \frac{1}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)}\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta_i+2\theta_t) - \frac{1}{2}\cos(2\theta_i-2\theta_t) - \frac{1}{2}\cos(2\theta_i-2\theta_t)\cdot\cos(2\theta_i-2\theta_t) +\right.$$

$$\left.\cos(2\theta_i-2\theta_t) - (\cos(2\theta_i+2\theta_t))\right) = \frac{1}{2\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)}\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta_i-2\theta_t) - \frac{1}{2}\cos(2\theta_i+2\theta_t) - \frac{1}{2}\cos(2\theta_i-2\theta_t)\cos(2\theta_i-2\theta_t)\right) =$$

$$\frac{\frac{1}{4}}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)}\left((1-\cos(2\theta_i+2\theta_t))(1+\cos(2\theta_i-2\theta_t)\right) = \frac{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)} = 1$$

$R+T=1$?

3.4)

Brewster-en angelua $\rightarrow$ $\tan\theta_B = \frac{n_2}{n_1}$ $\longrightarrow$ $E_{rp}=0$

$$\frac{E_{rp}}{E_{ip}} = \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)} = 0 \rightarrow \quad \theta_i+\theta_t = \frac{\pi}{2} \quad \text{edo} \quad \theta_i-\theta_t = 0 \rightarrow \theta_i = \theta_t$$

- $\theta_i = \theta_t$ $\rightarrow$ Snell: $n_1 \sin\theta_i = n_2 \sin\theta_t \overset{\theta_i=\theta_t}{=} n_2 \sin\theta_i \rightarrow \sin\theta_i(n_1-n_2) = 0 \longleftrightarrow$

$n_1 \cancel{=} n_2$ edo $\theta_i = 0$

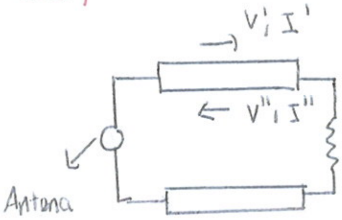
- $\theta_i+\theta_t=\frac{\pi}{2} \rightarrow$ Snell : $n_1 \sin\theta_i = n_2 \sin\theta_t = n_2 \sin\left(\frac{\pi}{2}-\theta_i\right) = n_2 \cos\theta_i \Rightarrow$

$\theta_t = \frac{\pi}{2}-\theta_i$

$$\boxed{\operatorname{tg}\theta_i = \frac{n_2}{n_i}}$$

## TRANSMISIO LERROAK:

3.5.)




- $V''(x,t) \Rightarrow$ ezkerrerantz
- $V'(x,t) \Rightarrow$ eskuinerantz.

$\Big\}\ V(x,t) = V''(x,t) + V'(x,t)$

$r = V''/V' = \Gamma$ (islapen koef.)   $t = V/V' = \tau$ (transmisio koef.)

- Erresistentzian neurketakoa $R = \frac{V_R}{I_R}$ (Ohm)

$$\begin{cases} V_R = V'+V'' \\ I_R = I'-I'' = \frac{1}{Z_0}(V'-V'') \end{cases}$$

$Z_0$ → transmisio braren inpedantzia karakteristikoa

$I'-I''$ → korronteen norabidea kontuan hartu behar da

$$* \ R = \frac{V_R}{I_R} = \frac{V'+V''}{I'-I''} = Z_0\frac{(V'+V'')}{(V'-V'')} = Z_0\frac{(1+V''/V')}{(1-V''/V')} = Z_0\left(\frac{1+r}{1-r}\right) \rightarrow$$

$r = V''/V'$

$$\frac{R}{Z_0}(1-r) = 1+r \rightarrow r\left(1+\frac{R}{Z_0}\right) = \frac{R}{Z_0}-1 \rightarrow r = \frac{R-Z_0}{R+Z_0}$$

$$\begin{cases} R\to 0 \rightarrow r \to -1 & \text{(zirkuitu laburra)} \\ R\to\infty \rightarrow r\to 1 \end{cases}$$

$r = V''/V'$   $t = V/V' = V_R/V'$

$$* \ R = \frac{V_R}{I_R} = \frac{V_R}{I'-I''} = Z_0\frac{V_R}{V'-V''} = Z_0\frac{V_R/V'}{1-V''/V'} = Z_0\frac{t}{1-r} \rightarrow \frac{R}{Z_0}(1-r) = t \rightarrow$$

$$t = \frac{R}{Z_0}\left(1-\frac{R-Z_0}{R+Z_0}\right) = \frac{R}{Z_0}\left(\frac{\cancel{R}+Z_0\cancel{-R}+Z_0}{R+Z_0}\right) = \frac{2R}{R+Z_0}$$

$$\begin{cases} R\to 0 \rightarrow t\to 0 \\ R\to\infty \rightarrow t\to 2 & (t\leq 1 \text{ energia}) \end{cases}$$

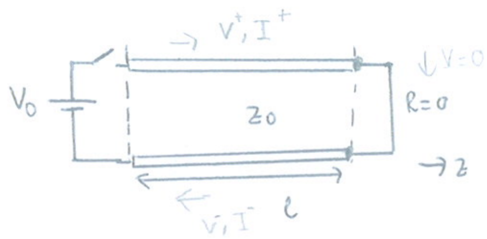
Xahutzen den potentzia : $P = I_R^2 R = I_R^{\cancel{2}}\frac{V_R}{\cancel{I_R}} = I_R V_R = (I'-I'')(V'+V'') = \frac{1}{Z_0}(V'-V'')(V'+V'') =$

$$\frac{1}{Z_0}\left(V'^2 - V''^2\right) = \frac{1}{Z_0}\left(1 - \frac{V''^2}{V'^2}\right) = \frac{1}{Z_0}(1-r) \rightarrow \text{Maximoa} \quad r=0 \rightarrow r = \frac{R-Z_0}{R+Z_0} = 0 \rightarrow$$

$$\boxed{R = Z_0} \quad (t = 1 \rightarrow r + t = 1)$$

3.6)

$Z_0$ bereko inpedantzia → $v$ hedapen abiadura eta $l$ luzera, $T = l/v$ ← uhinak linea osoa zehar hedatzeko behar duen denbora



$t=0 \rightarrow V_0$ bateria konstanteaz elikatu.

Trans. linea muturrean kortozirkuituan lotuta dago $(R=0)$

$$Z_0 = \sqrt{\frac{L_0}{C_0}}$$

$$\begin{cases} V = V^+ + V^- = V(z,t) \\ I = I^+ + (-I^-) = I^+ - I^- = \frac{1}{Z_0}(V^+ - V^-) = I(z,t) \end{cases} \qquad \begin{cases} V(0,t) = V_0 \quad t \geq 0 \\ V(l,t) = 0 \end{cases}$$

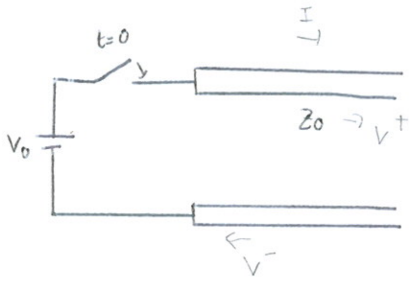
$z = l \rightarrow$ Kortozirkuituan $\rightarrow$ $V(l,t) = V^+ + V^- = 0 \rightarrow V^+(l-vt) = -V^-(l+vt)$

$$I(l,t) = \frac{1}{Z_0}(V^+ - V^-) = \frac{2V^+(l-vt)}{Z_0}$$

$z = 0 \rightarrow V(0,t) = V^+(-vt) + V^-(vt) = V_0 \rightarrow V^+(-vt) = V_0 - V^-(vt)$

$$I(0,t) = \frac{1}{Z_0}\left(V^+(-vt) - V^-(vt)\right) = \frac{(2V^+(-vt) - V_0)}{Z_0}$$

3.7.)

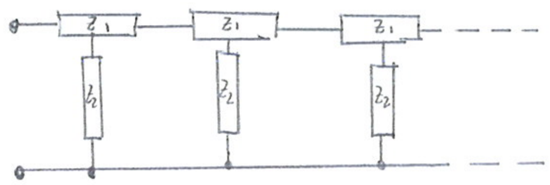



$V(0,t) = V_0$

$t=0 \rightarrow$ etengailua itxi

Demagun $Z_0$ dela transmisio lerroaren inpedantzia karakteristikoa.

3.8.)



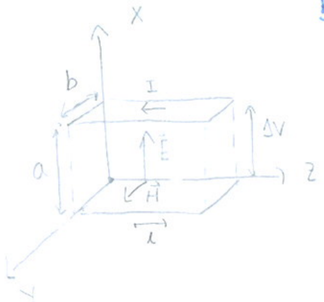

3.9.)

$Z_0$ inpedantzia karakteristikoa, $z=0 \rightarrow V(t) = V_0 \cos(wt)$ $\quad v = \frac{1}{\sqrt{C_0 L_0}}$

($v$ hedapen abiadura eta mugagabea), $V, I$?

$V = V^+ + V^- =$

3.10.)




b zabaleradun eta a distantziaz banatutako plano paraleloak (transmisio lerroa)

$$\begin{cases} E(z,t) = E_x = E_0 e^{-i(\omega t - Kz)} \\ H(z,t) = H_y = H_0 e^{-i(\omega t - Kz)} = E_0 \sqrt{\frac{\varepsilon}{\mu}}\, e^{-i(\omega t - Kz)} \end{cases}$$

- $\Delta V = \int_0^a E_x\, dx = E_0 a\, e^{-i(\omega t - Kz)}$

- $\vec{K} = \hat{n} \times \vec{H} \rightarrow I = K \cdot b = H_y\, b = b E_0 \sqrt{\frac{\varepsilon}{\mu}}\, e^{-i(\omega t - Kz)}$

  Ampère

  planoetan bakarrik

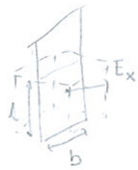
- $Z_0 = \frac{\Delta V}{I} = \frac{a\, \cancel{E_0 e^{-i(\omega t - Kz)}}}{b \sqrt{\varepsilon/\mu}\, \cancel{E_0 e^{-i(\omega t - Kz)}}} = \sqrt{\frac{\mu}{\varepsilon}} \left(\frac{a}{b}\right) = \sqrt{\frac{L_0}{C_0}} \rightarrow L_0 = Z_0^2 C_0$

- $C_0 = \frac{C}{l} = \frac{Q}{\Delta V l} = \frac{Q/l}{\Delta V} = \frac{Q/l}{E_x a} = \frac{E_x b \varepsilon}{E_x a}$

- $v = \frac{1}{\sqrt{L_0 C_0}} = \frac{1}{Z_0 C_0} = \frac{\cancel{b}\sqrt{\varepsilon}}{\cancel{a}\sqrt{\mu}} \cdot \frac{\cancel{a}}{\cancel{b}\varepsilon} = \frac{1}{\sqrt{\mu \varepsilon}}$

  dielektrikoaren abiadura bera.

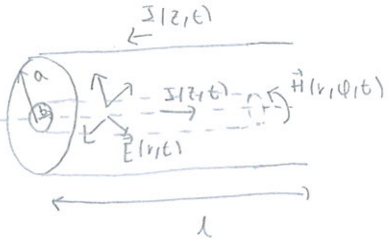
Gauss $\rightarrow E_x\, b \cdot l = \frac{Q}{\varepsilon} \rightarrow E_x b \varepsilon = \frac{Q}{l}$




$\int \vec{E}\, dA = E_x (b \cdot l)$

(Gauss)

3.11.)




Koordenatu zilindrikoak : $(r, \varphi, z)$ $(a > b)$

- $\vec{E}(r,z,t) = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}\, e^{-i(Kz - \omega t)} = \lambda(z,t) \frac{\hat{r}}{2\pi\varepsilon r}$ (Gauss) $r > b$

- $\underset{V(a)-V(b)}{\Delta V} = \int_b^a -\vec{E} \cdot d\vec{r} = \frac{\lambda}{2\pi\varepsilon_0} e^{-i(Kz - \omega t)} \ln\left(\frac{b}{a}\right) = \frac{\lambda(z,t)}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right) =$

  $\Delta V(z,t)$

- $C = \frac{Q}{\Delta V} \Rightarrow C_0 = \frac{C}{l} = \frac{Q/l}{\Delta V} = \frac{\lambda(z,t)}{\Delta V} = \frac{2\pi\varepsilon}{|\ln(b/a)|}$


$ds = l \cdot dr$

- $\underset{b<r<a}{\vec{H}(r,t)} = \frac{I}{2\pi r} e^{-i(Kz - \omega t)}\, \hat{u}_\varphi = \frac{I(z,t)}{2\pi r} \hat{u}_\varphi$ (Ampère) $\Rightarrow \Phi_{Mag} = \iint_S \vec{B}_s\, d\vec{s} = \mu \frac{I(z,t)\, l \ln\left(\frac{a}{b}\right)}{2\pi}$

- $L_0 = L/l = \frac{\Phi_{mag}(z,t)/I(z,t)}{l} = \frac{\mu \cancel{l} \ln\left(\frac{a}{b}\right)}{2\pi \cancel{l}} = \frac{\mu}{2\pi}\left|\ln\left(\frac{b}{a}\right)\right| = \frac{\mu}{2\pi}\ln\left(\frac{a}{b}\right)$

- $v = \frac{1}{\sqrt{L_0 C_0}} \overset{*}{=} \frac{1}{\sqrt{\mu\varepsilon}}$
- $Z_0 = \sqrt{\frac{L_0}{C_0}} = \sqrt{\frac{\frac{\mu}{2\pi}\ln(b/a)}{(2\pi\varepsilon/\ln(b/a))}} = \underbrace{\ln\left(\frac{b}{a}\right)}_{\text{geometriaren menpekoa}} \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{1}{2\pi}$

$* \; L_0 C_0 = \frac{\cancel{2\pi}\varepsilon}{\cancel{\ln(b/a)}} \cdot \frac{\mu}{\cancel{2\pi}} \cancel{\ln\left(\frac{b}{a}\right)}$

3.12.)

Hari paraleloko transmisio lerroa $\Rightarrow$ $L_0 = \frac{\mu_0}{\pi}\ln\left(\frac{d}{a}\right)$ { $d$ harien arteko distantzia; $a << d$ harien erradioa

$v = \frac{1}{\sqrt{\mu\varepsilon}}$ (seinaleen hedapen abiadura) $C_0$?

$$v = \frac{1}{\sqrt{L_0 C_0}} \rightarrow C_0 = \frac{1}{L_0 v^2} = \frac{\mu\varepsilon}{\frac{\mu_0}{\pi}\ln\left(\frac{d}{a}\right)} \underset{\text{suposatu } \mu \simeq \mu_0}{=} \frac{\pi\varepsilon}{\ln\left(\frac{d}{a}\right)}$$

3.13.)

Kable-ardazkide koaxiala $\rightarrow$ $a = 0'4$ mm, $b = 2'5$ mm $\varepsilon_r = 2'26$ (dielektrikoa haien artean)

a) 3.11-ko emaitzak erabiliz: $v = \frac{1}{\sqrt{\mu\varepsilon}} \underset{\text{suposatu } \mu=\mu_0}{=} \frac{1}{\sqrt{\mu_0\varepsilon_r\varepsilon_0}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \cdot \frac{1}{\sqrt{\varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}} = 1'99 \cdot 10^8$ m/s

$$Z_0 = \sqrt{\frac{L_0}{C_0}} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{2\pi}\left|\ln\left(\frac{a}{b}\right)\right| \underset{\mu=\mu_0}{=} \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}} \cdot \frac{1}{2\pi}\left|\ln\left(\frac{b}{a}\right)\right| = 73'107\ \Omega$$

b) Dielektriko arrunten permitibitate erlatiboa, $\varepsilon_r \in [2, 5]$

$$v = \frac{c}{\sqrt{\varepsilon_r}} \in [1'34 \cdot 10^8 \text{ m/s}, 2'12 \cdot 10^8 \text{ m/s}]$$

$$Z_0 = \frac{1}{\sqrt{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{2\pi}\left|\ln\left(\frac{a}{b}\right)\right| \in [49'15\ \Omega, 77'71\ \Omega]$$

3.14.)

$a = 5\,\mu m$ (zabalera) $b = 1\,\mu m$ (lodiera) $\Rightarrow$ plano paraleloko transmisio lerroak.

$\varepsilon_r = 2'5$, $\mu_r = 1 \rightarrow \mu = \mu_0$ (dielektrikoz betetа)

3.10. ariketan emaitza:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}} = 1'897 \cdot 10^{8}\ m/s$$

$$Z_0 = \sqrt{\frac{\mu}{\varepsilon}}\left(\frac{b}{a}\right) = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{\sqrt{\varepsilon_r}}\left(\frac{b}{a}\right) = 47'66\ \Omega$$

Dielektrikoaren lodiera erdira jaitsi → $b = 0'5\ \mu m$

$$v = \frac{c}{\sqrt{\varepsilon_r}} = 1'897 \cdot 10^{8}\ m/s \qquad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{\sqrt{\varepsilon_r}}\left(\frac{b}{a}\right) = 23'83\ \Omega$$

3.15.)

Transmisio-lerroaki transizio denbora:

a) GaAs $(\varepsilon_r = 11)$ ⇒ xafla mehez egindako lerroa: $d = 100\ \mu m$ aldenduta dauden bi elementu lotu.

Abiadura: $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = 9 \cdot 10^{7}\ m/s$ $\qquad v = \frac{d}{\Delta t} \rightarrow \Delta t = \frac{d}{v} = \frac{100\ \mu m}{9 \cdot 10^{7}\ m/s} = 1'11 \cdot 10^{-12}\ s = 1'11\ ps$

3.10 ariketa

b) Silizio $(\varepsilon_r = 12)$ ⇒ xafla mehez egindako lerroa: $d = 1\ mm$ aldenduta:

Abiadura: $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = 8'66 \cdot 10^{7}\ m/s$ $\qquad v = \frac{d}{\Delta t} \rightarrow \Delta t = \frac{d}{v} = \frac{1\ mm}{8'66 \cdot 10^{7}\ m/s} = 1'15 \cdot 10^{-11}\ s = 11'5\ ps$

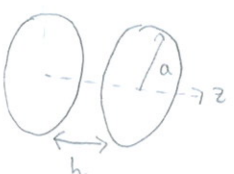
c) $l = 100\ m$ luzerako kable-ardazkidea, $\varepsilon_r = 2'5$.

Abiadura: $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = 1'89 \cdot 10^{8}\ m/s$ $\qquad v = \frac{l}{\Delta t} \rightarrow \Delta t = \frac{l}{v} = \frac{100\ m}{1'89 \cdot 10^{8}\ m/s} = 5'27 \cdot 10^{-7}\ s = 0'527\ \mu s$

3.11 ariketa

3.16.)

h distantziaz banatutako eta a erradioko xafla biribilez osatutako kondentsadorea.



3.17.)

L induktantzia eta C kondentsadorea seriean elkartu → $w_0 = \frac{1}{\sqrt{LC}}$ denean sistema erresonantea.


$L\downarrow \rightarrow w_0\uparrow$

## UHIN GIDAK eta KABITATE ERRESONANTEAK

3.18.)

Espazio hutsean → $\mu = \mu_0$, $\varepsilon = \varepsilon_0$ → $\vec{E} = E_0 \sin(Kz - wt)\,\hat{j}$  $(E_0 = 1000\ V/m)$

$f = 9\ GHz = 9\cdot 10^9\ Hz \rightarrow w = 2\pi f = 18\pi\cdot 10^9\ rad/s$

z norabidean (positiboan) hedatzen da uhina

a) $v = \frac{1}{\sqrt{\mu\varepsilon}} = c = \frac{w}{K} \rightarrow K = \frac{w}{c} = \frac{18\pi\cdot 10^9}{3\cdot 10^8}\ m^{-1} = 60\pi\ m^{-1} \Rightarrow \hat{K} = 60\pi\,\hat{k}\ m^{-1}$

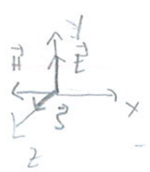
$\lambda = \frac{2\pi}{K} = \frac{2\pi}{60\pi}\ m = \frac{1}{30}\ m = \frac{c}{f}$

$\vec{H} = \hat{k} \times \frac{\vec{E}}{c\mu_0} = \frac{1}{\mu_0}\hat{k}\times\frac{\vec{E}}{c} = -\frac{1}{\mu_0 c} E_0 \sin(Kz - wt)\,\hat{\imath} = -\frac{\sqrt{\mu_0\varepsilon_0}}{\mu_0} E_0 \sin(Kz - wt)\,\hat{\imath} = -\sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \sin(Kz - wt)\,\hat{\imath} =$

$-2'653 \sin(Kz - wt)\,\hat{\imath}$  $(A/m)$

$\vec{S} = \vec{E}\times\vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2 \sin^2(Kz - wt)\,\hat{k} = 2653'79 \sin^2(Kz - wt)\,\hat{k}$  $(W/m^2)$

b) Maiztasun horretako uhina ($\nu = 9\cdot 10^9\ Hz$) → $a = 3$ cm-ko gidan barrena heda daiteke?




Heda dadin: $K_g^2 = K_0^2 - K_c^2 \geq 0$ izan behar da

$K_0 = \frac{w}{c} = 60\pi\ m^{-1}$  $K_c = \frac{n\pi}{a} = \frac{100 n\pi}{3}\ m^{-1}$

$K_0^2 = 3600\pi^2$  $K_c^2 = \frac{10000}{9} n^2\pi^2$  $n \in \mathbb{N}$

$K_0^2 - K_c^2 = n^2 \left( 3600 - \frac{10000}{9} n^2 \right) > 0 \rightarrow \quad 3600 > \frac{10000}{9} n^2 \rightarrow . \ n^2 < 3'24 \rightarrow n < 1'8$

Beraz, gehienez $n=1$ izan daiteke, bestela ez da hedatuko $\Rightarrow$ 1. modoa bano ez da hedatuko.

- Gida batean hedatzen bada: $\vec{E} = \overbrace{E_0 \sin\left(\frac{n\pi}{a}x\right)}^{E_0(x)} \sin(K_g z - wt)\hat{j} \rightarrow$ 1. modoa baino ezin da hedatu $\rightarrow n=1 \rightarrow E_0(x) = E_0 \sin\left(\frac{\pi}{a}x\right)$ ; $K_g = \sqrt{K_0^2 - K_c^2} = \sqrt{\frac{w^2}{c^2} - K_c^2} = \pi \cdot 49'89 \ m^{-1}$

$K_c = \frac{\pi}{a}$

- $\lambda_g = \frac{2\pi}{K_g} = \frac{2}{49'89} m = 0'0409 m$ eta $\lambda_0 = \frac{2\pi}{K_0} = \frac{1}{30} m$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} = \frac{\partial E_y}{\partial x}\hat{k} - \frac{\partial E_y}{\partial z}\hat{i} = E_0 \frac{\pi}{a} \cos\left(\frac{\pi}{a}x\right) \sin(K_g z - wt)\hat{k} +$

$- E_0 \sin\left(\frac{\pi}{a}x\right) K_g \cos(K_g z - wt)\hat{i}$ $\quad *^1$ (13. orr.)

3.19)

$a \times b$ uhin gida errektangularra, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0$ dielektrikoz beteta.

($a$: zabalera, $b$: altuera)

- Hutsik dagoen gida: $w_{mn}$ (ebaki maiztasuna) $= c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

Gidatuko uhin-luzera: $\lambda_g$ ? $\quad K_g^2 = K^2 - \frac{w_{mn}^2}{c^2} = \frac{w^2}{c^2} - \frac{w_{mn}^2}{c^2} \rightarrow$

$K_g = \frac{1}{c}\sqrt{w^2 - w_{mn}^2} = \frac{2\pi}{\lambda_g} \rightarrow * \ \lambda_g = 2\pi c \cdot \frac{1}{\sqrt{w^2 - w_{mn}^2}}$

$* \ v_f = \frac{w}{K_g} = \frac{c \cdot w}{\sqrt{w^2 - w_{mn}^2}}$ $\quad * \ v_g = \frac{dw}{dK_g} \overset{*}{=} \frac{1}{\cancel{2}} \cdot \cancel{2} K_g c^2 (K_g^2 c^2 + w_{mn}^2)^{-1/2} =$

$\frac{K_g c^2}{\sqrt{K_g^2 c^2 + w_{mn}^2}} = \frac{\sqrt{w^2 - w_{mn}^2}}{w} c$

$* \ \sqrt{K_g^2 c^2 + w_{mn}^2} = w$

$v_f \cdot v_g = c^2$

- Dielektrikoz beteta: $K_0^2 = K_g^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \rightarrow K_g = 0 \rightarrow K_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = K$

$v = \frac{w}{K} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0}} = \frac{c}{\sqrt{\varepsilon_r}} \rightarrow w_{mn} = \frac{c}{\sqrt{\varepsilon_r}} K_{mn} = \frac{c}{\sqrt{\varepsilon_r}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ (ebaki maiztasuna)

$$\left(\frac{2\pi}{\lambda_g}\right)^2 = \left(\frac{2\pi}{\lambda_0}\right)^2 - \left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2 = k_g^2 \rightarrow \quad k_g = \sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2} = \frac{2\pi}{\lambda_g} \rightarrow$$

$$* \lambda_g = \frac{2\pi}{\sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2}} = \frac{2\pi}{\sqrt{\frac{\omega^2\varepsilon_r}{c^2} - \omega_{mn}^2\frac{\varepsilon_r}{c^2}}} = \frac{2\pi c}{\sqrt{\varepsilon_r}\sqrt{\omega^2 - \omega_{mn}^2}}$$

$$k = \frac{\omega\sqrt{\varepsilon_r}}{c} \quad ; \quad \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2 = \omega_{mn}^2 \frac{\varepsilon_r}{c^2}$$

$$* v_f = \frac{\omega}{k_g} \rightarrow \frac{\omega}{\sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2}} = \frac{\omega}{\sqrt{\frac{\omega^2\varepsilon_r}{c^2} - \frac{\omega_{mn}^2\varepsilon_r}{c^2}}} = \frac{\omega \cdot c}{\sqrt{\varepsilon_r}\sqrt{\omega^2 - \omega_{mn}^2}}$$

$$* v_g = \frac{d\omega}{dk_g} = \frac{1}{\cancel{2}} \cancel{2} k_g \frac{c^2}{\varepsilon_r}\left(\frac{k_g^2 c^2}{\varepsilon_r} + \omega_{mn}^2\right)^{-1/2} = \frac{k_g c^2}{\varepsilon_r \cdot \omega} = \frac{1}{c} \frac{\sqrt{\omega^2 - \omega_{mn}^2} \cdot c^2}{\sqrt{\varepsilon_r}\ \omega}$$

$$* k_g = \sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2} = \frac{1}{c}\sqrt{\varepsilon_r}\sqrt{\omega^2 - \omega_{mn}^2} \rightarrow \sqrt{\frac{k_g^2 c^2}{\varepsilon_r} + \omega_{mn}^2} = \omega$$

$$v_f \cdot v_g = \frac{c\cancel{\sqrt{\omega^2 - \omega_{mn}^2}}}{\sqrt{\varepsilon_r} \cdot \cancel{\omega}} \cdot \frac{\cancel{\omega} c}{\sqrt{\varepsilon_r}\cancel{\sqrt{\omega^2 - \omega_{mn}^2}}} = \frac{c^2}{\varepsilon_r} = v^2$$

uhinaren hedapen abiadura.

3.20.)

$\nu = 9\ GHz = 9 \cdot 10^9\ Hz$ -ko erradiazio elektromagnetikoa.

a) Heda dadin → maiztasun minimoa, ebaki maiztasuna $\omega_{mn} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

$$2\pi\nu = \omega \geq \omega_{mn} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \rightarrow \frac{4\cancel{\pi^2}\nu^2}{c^2} \geq \frac{m^2\cancel{\pi^2}}{a^2} + \frac{n^2\cancel{\pi^2}}{b^2} \rightarrow \frac{4\nu^2}{c^2} = 3600 \geq \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \ (m^{-2})$$

$m, n \in \mathbb{N}$

b) $E_y = E_0 \sin\left(\frac{2\pi x}{a}\right)$ modua heda ez dadin zein tarteten alda daiteke gidaren dimentsioa?

$$\begin{cases} m = 2 \\ n = 0 \end{cases} \rightarrow \omega \leq \omega_{20} \ ; \ 3600\ m^{-1} \leq \frac{4}{a^2} \rightarrow a \leq \sqrt{\frac{4}{3600}}\ m = \frac{1}{30}\ m$$

b edozein izan daiteke eta $a \in (0, 1/30\ m)$ tartean egon daiteke.

$\nu = 30\ GHz = 3\cdot 10^{10}\ Hz \rightarrow$ $E_y = E_0 \sin\left(\frac{2\pi x}{a}\right)$ bada hoda ez dadin

$(m=2,\ n=0) \rightarrow \cancel{2}\pi\nu = \omega \leq \omega_{20} = c\frac{2\pi}{a}$ izan behar da $\rightarrow$ $a \leq \frac{c}{\nu} = 0'01\,m$

Orduan, b edozein izan daiteke eta $a \in (0,\ 0'01\,m)$ tartean egon behar da.

* 13.18. ariketa)

- $\frac{\partial \vec{B}}{\partial t} = -E_0 \overset{K_c}{\overset{\shortparallel}{\left(\frac{\pi}{a}\right)}} \cos\left(\frac{\pi}{a}x\right) \sin(K_g z - \omega t)\hat{k} + E_0 \sin\left(\frac{\pi}{a}x\right) K_g \cos(K_g z - \omega t)\hat{\imath} \rightarrow$

$\vec{B} = -E_0 \frac{K_c}{\omega} \cos\left(\frac{\pi}{a}x\right) \cos(K_g z - \omega t)\hat{k} - E_0 \frac{K_g}{\omega} \sin\left(\frac{\pi}{a}x\right) \sin(K_g z - \omega t)\hat{\imath}$

- $\vec{H} = \frac{\vec{B}}{\mu} \underset{\substack{\downarrow \\ \text{Hutsean, } \mu = \mu_0}}{=} \frac{-E_0}{\mu_0 \omega} \left( K_c \cos\left(\frac{\pi}{a}x\right) \sin(K_g z - \omega t)\hat{k} + K_g \sin\left(\frac{\pi}{a}x\right) \cos(K_g z - \omega t)\hat{\imath} \right)$

- $\vec{S} = \vec{E} \times \vec{H} = \frac{E_0^2}{\mu_0 \omega} \left( -K_c \cos(K_c x) \sin(K_c x) \sin(K_g z - \omega t) \cos(K_g z - \omega t)\vec{\imath} + \right.$

$\left. K_g \sin^2(K_c x) \cos^2(K_g z - \omega t)\hat{k} \right) = \frac{E_0^2}{\mu_0 \omega} \left( -\frac{K_c}{4} \sin(2K_c x) \sin(2K_g z - 2\omega t)\hat{\imath} + K_g \sin^2(K_c x) \cos^2(K_g z - \omega t)\hat{k} \right)$

3.22.)

Ingurune sakabanatzailea $\rightarrow$ $\omega = \omega(K)$ (dispertsio erlazioa) $\rightarrow$ uhinaren hedapen-konstantea eta maiztasunaren artean erlazio lineala ez dagoenean $(\omega^2 \neq v^2 K^2)$

$a = 2'54\,cm$-ko zabalera duen gida errektangeluarra (suposatu $b = a$)

- $\left(\frac{2\pi}{\lambda_g}\right)^2 = K_g^2 = K_0^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 = \left(\frac{2\pi}{\lambda_0}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 = \frac{\omega^2}{c^2} - \frac{\omega_{mn}^2}{c^2}$ ebaki maiztasuna

- $\omega^2 = c^2 K_g^2 + \omega_{mn}^2 \rightarrow \omega = c\sqrt{K_g^2 + \frac{\omega_{mn}^2}{c^2}}$ , $K_g = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$

- $v_f = \frac{\omega}{K_g} = \frac{c\omega}{\sqrt{\omega^2 - \omega_{mn}^2}} > c$ , $v_T = \frac{d\omega}{dK_g} = \frac{\cancel{2} K_g c \cdot \frac{1}{\cancel{2}}}{\sqrt{K_g^2 + \frac{\omega_{mn}^2}{c^2}}} = \frac{\sqrt{\omega^2 - \omega_{mn}^2}}{\omega / c} = \frac{c\sqrt{\omega^2 - \omega_{mn}^2}}{\omega} < c$

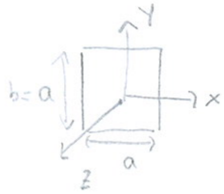
- $v_f \cdot v_T = c^2$ $\qquad \omega \rightarrow \omega_{mn} \left\{ \begin{array}{l} v_f \rightarrow \infty \\ v_T \rightarrow 0 \end{array} \right.$

Ad: $\omega_{10}\ (n=0,\ m=1) \rightarrow \quad \omega_{10} = c\sqrt{\left(\frac{\Pi}{a}\right)^2} = \frac{c\Pi}{a} = 3'71\cdot 10^{10}$ rad/s

$\omega_{20}\ (n=0,\ m=2) \rightarrow \quad \omega_{20} = c\,\frac{2\Pi}{a} = 7'42\cdot 10^{10}$ rad/s

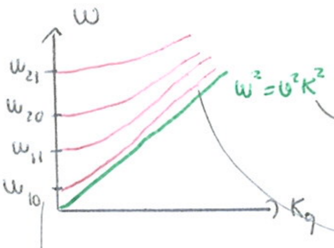
$\omega_{11}\ (n=m=1) \rightarrow \quad \omega_{11} = \frac{c\Pi}{a}\sqrt{2} = 5'24\cdot 10^{10}$ rad/s

$\hookrightarrow b=a$




$\omega_{21}\ (n=1,\ m=2) \rightarrow \omega_{21} = \frac{c\Pi}{a}\sqrt{5} = 8'297\cdot 10^{10}$ rad/s

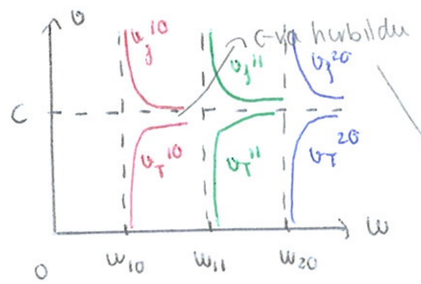
$\hookrightarrow b=0$




$\omega^2 = v^2K^2 \rightarrow$ dispertsio gabeko ingurunean

$\rightarrow$ hurbildu $\rightarrow$ $\omega_{mn}$ zenbat eta txikiagoa izan gehiago hurbildu

$\rightarrow K_g = 0$ denean $\omega = \omega_{mn}$




$\omega \rightarrow \omega_{mn} \begin{cases} v_g \rightarrow \infty \\ v_T \rightarrow 0 \end{cases}$

$\frac{\omega_{mn}}{\omega}$ txikiagoa joango delako $\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2} \rightarrow 1$

3.23)

a zabalerako gida errektangeluarra $\Rightarrow$ $TE_{10}$ modua hedatzen denean.

$TE_{10} \rightarrow m=1,\ n=0 \rightarrow \quad K_0^2 = K_g^2 + \left(\frac{n\Pi}{b}\right)^2 + \left(\frac{m\Pi}{a}\right)^2 \underset{m=1,\ n=0}{=} K_g^2 + \left(\frac{\Pi}{a}\right)^2 = \frac{\omega^2}{c^2} \rightarrow K_g = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{\Pi}{a}\right)^2}$

(Hedatzen bada $\frac{\omega^2}{c^2} > \left(\frac{\Pi}{a}\right)^2$ )

- $H_z(x,y,z) = A\cos\left(\frac{m\Pi}{a}x\right)\cos\left(\frac{n\Pi}{a}y\right)e^{K_g z i} = A\cos\left(\frac{\Pi}{a}x\right)e^{iK_g z}$

- $E_Y = -\frac{\mu_0\omega}{K_g}\left(K_g i - i\frac{\varepsilon_0\mu_0\omega^2}{K_g}\right)^{-1}\frac{\partial H_z}{\partial x} = +\frac{\mu_0\omega}{K_g}\left(K_g i - i\frac{\varepsilon_0\mu_0\omega^2}{K_g}\right)^{-1}\left(+A\frac{\Pi}{a}\sin\left(\frac{\Pi}{a}x\right)e^{iK_g z}\right) =$

$$\frac{-\mu_0\omega i}{K_g\left(K_g - \frac{\omega^2}{c^2K_g}\right)}\left(A\frac{\Pi}{a}\sin\left(\frac{\Pi}{a}x\right)\right)e^{iK_g z} = \frac{-\mu_0\omega}{K_g\left(K_g-\frac{\omega^2}{c^2K_g}\right)}A\left(\frac{\Pi}{a}\right)\sin\left(\frac{\Pi}{a}x\right)e^{i\left(K_g z + \frac{\Pi}{2}\right)}$$

- $E_x = \frac{\mu_0\omega}{K_g}\cdot\frac{1}{\left(K_g i - i\frac{\omega^2}{c^2K_g}\right)}\frac{\partial H_z}{\partial y} = 0$ ; $E_z = 0$ ($TE_{10}$ da)

$$E_y = -\frac{\mu_0 \omega \lambda_g}{2\pi} H_x = -\frac{\mu_0 \omega}{K_g} H_x \rightarrow H_x = \frac{-K_g}{\mu_0 \omega} E_y = +\frac{\cancel{\mu_0\omega}}{\cancel{K_g}} \cdot \frac{\cancel{K_g}}{\cancel{\mu_0\omega}} \frac{i}{K_g} \cdot \frac{\left(A\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\right)}{\left(K_g - \frac{\omega^2}{c^2 K_g}\right)} e^{iK_g z} =$$

$$+\frac{i}{K_g\left(K_g - \frac{\omega^2}{c^2 K_g}\right)} A\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right) e^{iK_g z} = \frac{1}{K_g\left(K_g - \frac{\omega^2}{c^2 K_g}\right)} A\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right) e^{i\left(K_g z + \frac{\pi}{2}\right)}$$

$$E_x = \frac{\mu_0 \omega \lambda_g}{2\pi} H_y = 0 \rightarrow H_y = 0$$

$$\frac{\omega_{mn}}{c} = \frac{\omega_{10}}{c} = \frac{\pi}{a}$$

$$* \quad K_g - \frac{\omega^2}{c^2 K_g} = K_g - \frac{1}{K_g}\left(K_g^2 + \frac{\omega_{mn}^2}{c^2}\right) = \cancel{K_g} - \cancel{K_g} - \frac{\omega_{mn}^2}{K_g c^2} = -\frac{1}{K_g}\left(\frac{\pi}{a}\right)^2$$

- $$E_y = \frac{+\mu_0 \omega}{+\left(\frac{\pi}{a}\right)^2} \cdot A\cancel{\left(\frac{\pi}{a}\right)}\sin\left(\frac{\pi}{a}x\right) e^{i\left(K_g z + \frac{\pi}{2}\right)} = E_0 \sin\left(\frac{\pi}{a}x\right) e^{i\left(K_g z + \frac{\pi}{2}\right)}$$

- $$H_x = -\frac{K_g}{\mu_0 \omega} E_y = \frac{-K_g A \sin\left(\frac{\pi}{a}x\right)}{\left(\frac{\pi}{a}\right)} e^{i\left(K_g z + \frac{\pi}{2}\right)} = -E_0 \frac{K_g}{\mu_0 \omega} \sin\left(\frac{\pi}{a}x\right) e^{i\left(K_g z + \frac{\pi}{2}\right)}$$

- $$H_z = \frac{\pi}{\omega a \mu_0} E_0 \cos\left(\frac{\pi}{a}x\right) e^{iK_g z} \rightarrow E_y(t) = E_y e^{-i\omega t}, \quad H_z(t) = H_z e^{-i\omega t}$$

- $$\vec{E}(x,y,z,t) = E_y(t)\,\hat{j}\,;\ \vec{H}(x,y,z,t) = H_z(t)\hat{k} + H_x(t)\hat{i} \rightarrow \vec{S} = \vec{E} \times \vec{H} = -E_y H_x \hat{k} + E_y H_z \hat{i} =$$

$$\left(\cancel{E_0^2 \frac{K_g}{\mu_0 \omega} \sin^2\left(\frac{\pi}{a}x\right) e^{2i\left(K_g z + \frac{\pi}{2} - \omega t\right)} \hat{k} + E_0^2 \frac{\pi/a}{\mu_0 \omega} \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{a}x\right) e^{i\left(2K_g z + \frac{\pi}{2} - 2\omega t\right)} \hat{i}} = \right.$$

$$\left. \cancel{-E_0^2 \frac{K_g}{\mu_0 \omega} \sin^2\left(\frac{\pi}{a}x\right) e^{2i\left(K_g z - \omega t\right)} \hat{k} + E_0^2 \frac{(\pi/a)}{\mu_0 \omega}\frac{1}{2}\sin\left(\frac{2\pi}{a}x\right) e^{i\left(2K_g + \frac{\pi}{2} - 2\omega t\right)} \hat{i}} \right)$$

- $$v_g = \frac{\omega}{K_g} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \left(\frac{\pi}{a}\right)^2}} = \frac{\omega c}{\sqrt{\omega^2 - c^2\left(\frac{\pi}{a}\right)^2}} = \frac{\omega c}{\sqrt{\omega^2 - \omega_{10}^2}} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{10}}{\omega}\right)^2}} > c$$

- $$v_T = \frac{d\omega}{dK_g} = \frac{cK_g}{\omega/c} = \frac{c^2 K_g}{\omega} = \frac{c^2\sqrt{\frac{\omega^2}{c^2} - \left(\frac{\pi}{a}\right)^2}}{\omega} = c\sqrt{1 - \left(\frac{\omega_{10}}{\omega}\right)^2} < c$$

$$\omega = c\sqrt{K_g^2 + \left(\frac{\pi}{a}\right)^2}$$

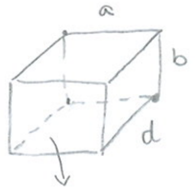
$$\vec{S} = Re(\vec{E}) \times Re(\vec{H}) = E_0 \sin\left(\frac{\pi}{a}x\right)\sin(K_g z - \omega t)\,\hat{j} \times \left[-E_0 \frac{K_g}{\mu_0 \omega} \sin\left(\frac{\pi}{a}x\right)\sin(K_g z - \omega t)\,\hat{i} + \right.$$

$$\left. \frac{\pi/a}{\omega \mu_0} E_0 \cos\left(\frac{\pi}{a}x\right)\cos(K_g z - \omega t)\,\hat{k}\right] = \frac{E_0^2}{\mu_0 \omega}\left(K_g \sin^2\left(\frac{\pi}{a}x\right)\sin^2(K_g z - \omega t)\,\hat{k} + \frac{(\pi/a)}{4}\sin\left(\frac{2\pi}{a}x\right)\sin(2K_g z - 2\omega t)\,\hat{i}\right)$$

$\langle S_z \rangle = \frac{1}{4} \frac{E_0^2 K_g}{\mu_0 \omega}$ $\qquad \langle S_x \rangle = 0$

3.24.)

$a = 3\,cm$, $b = 2\,cm$ eta $d = 4\,cm$




30 μm-ko xafla eroalea (lodiera)

Modorik baxuenaren maiztasuna $\rightarrow$ $\omega_{lmn} = c\sqrt{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{d}\right)^2}$

Gehienez bat izan daiteke nulua, bestela eremu guztiak zero direlako.

$b < a < d \Rightarrow$ minimoa $\rightarrow$ $\omega_{1,1,0} = c\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 5'663 \cdot 10^{10}$ rad/s

?

- 30 μm-ko xafla (lodiera) $\leftrightarrow$ $b = 2\,cm - 30 \cdot 10^{-4}\,cm = 1'997\,cm$

  $\searrow$ eroaleetan islatu egiten da guztia

  $\omega_{1,1,0} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 5'669 \cdot 10^{10}$ rad/s

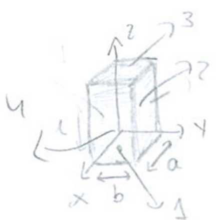
- ab aurpegian $\rightarrow$ $d = 4\,cm - 30 \cdot 10^{-4}\,cm = 3'997\,cm \rightarrow \omega_{1,1,0} = 5'663 \cdot 10^{10}$ rad/s

  (ez da aldatzen)

- db aurpegian $\rightarrow$ $a = 3\,cm - 30 \cdot 10^{-4}\,cm = 2'997\,cm \rightarrow \omega_{1,1,0} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 5'665 \cdot 10^{10}$ rad/s

3.25.)

$TE_{10}$ modoa $\rightarrow$ $n = 0$, $m = 1$ $\qquad \omega > \frac{\pi}{a} c = \omega_{10}$ $\qquad$ 3.23 ariketa




$B_x = -\frac{K_g}{\omega} E_0 \sin\left(\frac{\pi}{a}x\right) \sin(K_g z - \omega t)$, $E_x = 0$, $E_y = E_0 \sin\left(\frac{\pi}{a}x\right) \sin(K_g z - \omega t)$

$B_y = 0$, $B_z = \frac{\pi/a}{\omega} E_0 \cos\left(\frac{\pi}{a}x\right) \cos(K_g z - \omega t)$

- Eremu magnetikoaren osagai tangentzialak $\rightarrow$ gainazaleko korronteak.

Poynting bektorea $\Rightarrow$ $\vec{S} = \vec{E} \times \vec{H} = \frac{E_0^2}{\mu_0 \omega}\left( K_g \sin^2\left(\frac{\pi}{a}x\right) \sin^2(K_g z - \omega t)\hat{k} + \frac{(\pi/a)}{4} \cdot \sin\left(\frac{2\pi}{a}x\right) \cdot \right.$
$\left. \sin(2K_g z - 2\omega t)\hat{i} \right)$

$\nearrow I = K \cdot l$

Korronteak: $\hat{n} \times \hat{H} = \vec{K}$

- 2. aurpegia: $\hat{n} = +\hat{\jmath} \rightarrow \vec{K}_2 = \frac{1}{\mu_0}(\hat{n} \times \vec{B})|_{y=b}$
- 4. aurpegia: $\hat{n} = -\hat{\imath}$
- 3. aurpegia: $\hat{n} = -\hat{k}$
- 1. aurpegia: $\hat{n} = \hat{k}$

# ELEKTROMAGNETISMOA II: UHIN ELEKTROMAGNETIKOAK MUGARIK GABEKO INGURUNETAN

16-10-20

2.1)

$D_i = \sum_j \varepsilon_{ij} E_j$ , $B_i = \sum_j \mu_{ij} H_j$ ($\varepsilon_{ij}$, $\mu_{ij}$ tentsore simetriko eta konstanteak)

$$* \ \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \sum_i E_i \frac{\partial D_i}{\partial t} = \sum_i E_i \frac{\partial}{\partial t}\left(\sum_j \varepsilon_{ij} E_j\right) = \sum_{i,j} \varepsilon_{ij} E_i \frac{\partial E_j}{\partial t} = \sum_{i,j} \varepsilon_{ij} \frac{\partial}{\partial t}\left(\frac{1}{2} E_i E_j\right) = \sum_{ij} \frac{\partial}{\partial t}\left(\varepsilon_{ij} E_i E_j \cdot \frac{1}{2}\right)$$

$$\sum_i \frac{\partial}{\partial t}\left(\frac{1}{2}\sum_j E_i \varepsilon_{ij} E_j\right) = \sum_i \frac{\partial}{\partial t}\left(\frac{1}{2} E_i \sum_j \varepsilon_{ij} E_j\right) = \sum_i \frac{\partial}{\partial t}\left(\frac{1}{2} E_i D_i\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D}\cdot\vec{E}\right)$$

$$* \ \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \sum_i H_i \frac{\partial B_i}{\partial t} = \sum_i H_i \frac{\partial}{\partial t}\left(\sum_j \mu_{ij} H_j\right) = \sum_{i,j} \mu_{ij} H_i \frac{\partial}{\partial t} H_j = \sum_{i,j} \mu_{ij} \frac{1}{2}\frac{\partial}{\partial t}(H_i H_j) =$$

$$\sum_i \frac{1}{2}\frac{\partial}{\partial t}\left(\sum_j H_i H_j \mu_{ij}\right) = \sum_i \frac{1}{2}\frac{\partial}{\partial t}\left(H_i \sum_j H_j \mu_{ij}\right) = \sum_i \frac{1}{2}\frac{\partial}{\partial t}(H_i B_i) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{B}) = \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H}\cdot\vec{B}\right)$$

2.2)

adierazpen dipolarra

$\vec{m} = m\hat{k}$ , $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$ , $\vec{H} = \frac{1}{4\pi}\left[\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]$

$\vec{r} = x\vec{i} + y\vec{j} + z\hat{k}$

$r = \sqrt{x^2+y^2+z^2}$

$$\vec{S} = \vec{E}\times\vec{H} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} \times \frac{1}{4\pi r^3}\left[\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right] = \frac{q}{16\pi^2\varepsilon_0 r^6}\left(\cancel{\vec{r}\times\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^2}} - \vec{r}\times\vec{m}\right) =$$

$$\frac{q}{16\pi^2\varepsilon_0 r^6}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -x & -y & -z \\ 0 & 0 & m \end{vmatrix} = \frac{qm}{16\pi^2\varepsilon_0 r^6}\left(-y\vec{i} + x\vec{j}\right) = \frac{qm}{16\pi^2\varepsilon_0}\frac{(x\vec{j} - y\vec{i})}{(x^2+y^2+z^2)^3}$$

$$\nabla\cdot\vec{S} = \nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H}) = 0$$

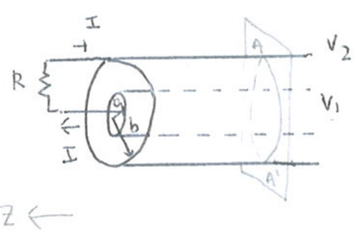
$*$

$*$ Elektrostatika: $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = 0$ ; $\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} = 0$

$\downarrow$

$\vec{J} = 0$ , $\frac{\partial\vec{D}}{\partial t} = 0$

2.3.)




a)

$(V_1 > V_2)$ $\quad a < r < b \rightarrow \oint \vec{H}\cdot d\vec{l} = I = H\cdot 2\pi r = \frac{B}{\mu_0} 2\pi r \rightarrow \vec{B} = \frac{I\mu_0}{2\pi r}\hat{\varphi}$

$r < a \rightarrow \oint \vec{H}\cdot d\vec{l} = \oint \vec{J}\, d\vec{a} = I\left(\frac{r}{a}\right)^2;\quad r > b \rightarrow B = 0$ (Ampere)

$H = \frac{I r^2}{2\pi r a^2}$

Gauss $\rightarrow$ $E\cdot 2\pi r \cancel{l} = \frac{Q_{in}}{\varepsilon_0} = \frac{\lambda \cancel{l}}{\varepsilon_0} \rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$ $\quad (a<r<b) \rightarrow \vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} \rightarrow$

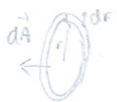
$\int_{V_1}^{V_2} \partial V = -\int_a^b \frac{\lambda}{2\pi\varepsilon_0 r} dr = V_2 - V_1 = -\frac{\lambda}{2\pi\varepsilon_0} \ln r \Big]_a^b = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{a}{b}\right) \rightarrow \frac{\lambda}{2\pi\varepsilon_0} = \frac{V_2 - V_1}{\ln\left(\frac{a}{b}\right)}$ $\rightarrow$ ($\lambda$ ez dakigu dan bat)

$\vec{E} = \frac{V_2 - V_1}{\ln\left(\frac{a}{b}\right)} \frac{\hat{r}}{r}$ ; $\quad \vec{S} = \vec{E}\times\vec{H} = \frac{V_2 - V_1}{2\pi r^2} \frac{I}{\ln\left(\frac{a}{b}\right)}\hat{k}$ $\quad (a<r<b)$

↳ hemendik kanpo 0

b)

$P = \iint_S \vec{S}\cdot d\vec{A} = \int_a^b \frac{(V_2-V_1)I}{\ln\left(\frac{a}{b}\right)} \frac{\cancel{2\pi} r\, dr}{\cancel{2\pi} r^2} = \frac{(V_2-V_1)I}{\ln\left(\frac{a}{b}\right)} \int_a^b \frac{dr}{r} = \frac{(V_2-V_1)I}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{b}{a}\right) = I(V_1 - V_2)$

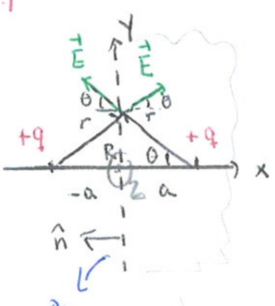



$dA = 2\pi r dr$

$\Rightarrow$ Berdina da

Erresistentzian iraungitutako potentzia: $P = I^2 R = I^2 \underbrace{\frac{(V_1 - V_2)}{I}}_{\Delta V} = I(V_1 - V_2)$

2.4.)




ZY planoan $\nearrow$ karga gainean indarra $\rightarrow$ politikoa inguratzen duen eduki bolumena, integratu

$\vec{F}_i = \left(\frac{d\vec{p}}{dt}\right)_i = \sum_j \oiint_S T_{ij} n_j ds$ $\quad (j = 1,2,3 \quad i = 1,2,3)$ $\quad *$

$\hat{n} = n_x \vec{i} + n_y \vec{j} + n_z \hat{k}$ $\quad |\hat{n}| = 1$ , gure kasuan $\rightarrow \hat{n} = -\hat{i}$ $\quad (n_y = n_z = 0)$

$E_T = 2E\sin\theta = 2\cdot\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cdot\sin\theta \rightarrow \vec{E} = E_T\vec{j} = E_y\vec{j}$

↳ x-ren osagaiak anulatu

gure kasua (2 narraduan)

$x \begin{cases} T_{xx} = T_{11} = \varepsilon_0\cdot\frac{1}{2}(-E_y^2) = -E_T^2\frac{\varepsilon_0}{2} \\ T_{xy} = T_{12} = \varepsilon_0\cdot(E_x E_y) = 0 \quad T_{xz} = T_{13} = E_x E_z \varepsilon_0 = 0 \end{cases}$

$y \begin{cases} T_{yx} = 0 = T_{21} \quad T_{yy} = T_{22} = \frac{1}{2}\varepsilon_0 E_y^2 = E_T^2\frac{\varepsilon_0}{2} \\ T_{yz} = T_{23} = 0 \end{cases}$ $\qquad z \begin{cases} T_{zx} = T_{31} = 0 \quad T_{zy} = T_{32} = 0 \\ T_{zz} = -\frac{1}{2}\varepsilon_0 E_y^2 = -\frac{\varepsilon_0}{2}E_T^2 \end{cases}$

$r = \sqrt{R^2 + a^2}$ , $\sin\theta = \frac{R}{\sqrt{R^2+a^2}}$

OYZ planoan marraztutako horia

$R = r\sin\theta = a\,tg\theta$ , $ds = 2\pi R dR = 2\pi R \frac{a}{\cos^2\theta} d\theta = 2\pi a^2 \frac{tg\theta}{\cos^2\theta} d\theta$

$$\vec{F}_x = \oint_S \varepsilon_0 \cdot \frac{1}{2} (-E_T^2) \cdot (-\vec{i}) dS = \frac{\varepsilon_0}{2} \oint_S E_T^2 \vec{i} dS = \frac{\varepsilon_0 \vec{i}}{2} \int_0^{\pi/2} \left( \frac{1}{2\pi\varepsilon_0} \frac{q}{r^2} \sin\theta \right)^2 2\pi a^2 \frac{tg\theta}{\cos^2\theta} d\theta =$$

$r = \frac{a}{\cos\theta}$

$$\frac{\varepsilon_0 q^2 \vec{i}}{2} \int_0^{\pi/2} \left( \frac{\cos^2\theta}{a^2} \frac{\sin\theta}{2\pi\varepsilon_0} \right)^2 2\pi a^2 \frac{tg\theta}{\cos^2\theta} d\theta = \cancel{\varepsilon_0} q^2 \vec{i} \int_0^{\pi/2} \frac{\cos^4\theta}{a^4} \frac{\sin^2\theta}{4\pi^2\varepsilon_0^2} \pi a^2 \frac{tg\theta}{\cos^2\theta} d\theta =$$

$$\frac{q^2}{(2a)^2 \varepsilon_0 \pi} \vec{i} \int_0^{\pi/2} \cos^2\theta \sin^2\theta \, tg\theta \, d\theta = \frac{q^2 \vec{i}}{\pi\varepsilon_0 (2a)^2} \int_0^{\pi/2} \cos\theta \sin^3\theta \, d\theta = \frac{q \vec{i}}{\pi\varepsilon_0 (2a)^2} \left. \frac{\sin^4\theta}{4} \right]_0^{\pi/2} =$$

$R = a \, tg\theta$

errodo bat daukt

$\geq 0$ izan behar da $\theta \in [0, \pi/2)$

distantzia bat da $R \Rightarrow$ positiboa

$$\frac{q \vec{i}}{4\pi\varepsilon_0 (2a)^2} \quad \text{(Coulomb)}$$

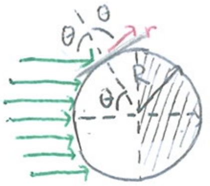
* edo $F_y = \frac{\varepsilon_0}{2} \int_0^{\infty} \left( \frac{1}{2\pi\varepsilon_0} \frac{q R}{(R^2+a^2)^{3/2}} \right)^2 2\pi R \, dR$

$$\vec{F}_y = \vec{F}_z = 0 \quad (n_x = n_z = 0)$$

2.5)

$u_{EM} = \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})$ (energia dentsitatea)

$r = \frac{\langle u_r \rangle}{\langle u_i \rangle}$




Energiaren $a$ zatia xurgatzen da eta $r = 1 - a$ islatzen da, $a = \frac{\langle u_t \rangle}{\langle u_i \rangle}$

a) Azalera unitateko indarra $z$ norabidean xurgapena dela eta: presioa.

xurg. $P_x(\theta) = \frac{\cos\theta}{c} (|\langle \vec{S}_i \rangle \cdot (-\hat{r})|) = \cos^2\theta \langle u_{EM} \rangle$

$r = \frac{\langle u_r \rangle}{\langle u_i \rangle} = \frac{\langle u_r \rangle}{\langle u_{EM} \rangle}$, $\frac{\langle \vec{S}_r \rangle}{c} = \langle u_r \rangle = r \langle u_{EM} \rangle$

b) islap. $P_i(\theta) = \frac{\cos\theta}{c} (|\langle \vec{S}_r \rangle \cdot (-\hat{r})|) = r \cos^2\theta \langle u_{EM} \rangle = (1-a) \cos^2\theta \langle u_{EM} \rangle$

Erradiazioa: $P_e = 2 u_{EM} \cos\theta$, Transmititutakoa: $P_t = a u_{EM} \cos\theta$

c) $P_T(\theta) = (1 + r) \cos^2\theta \langle u_{EM} \rangle = (1 + 1 - a) \cos^2\theta \langle u_{EM} \rangle = (2 - a) \cos^2\theta \langle u_{EM} \rangle$

$$F = \iint_A P_T(\theta) dA = \int_0^{\pi/2} R^2 2\pi \sin\theta \, d\theta \, (2-a) \cos^2\theta \langle u_{EM} \rangle = 2\pi R^2 (2-a) \langle u_{EM} \rangle \int_{-\pi/2}^{\pi/2} \sin\theta \cos^2\theta \, d\theta =$$

simetriaz

$dA = R \sin\theta \cdot 2\pi \cdot R d\theta$

$R\sin\theta$ zirkuluaren erradioa izan

$$2\pi R^2 (2-a) \langle u_{EM} \rangle \left( -\left. \frac{\cos^3\theta}{3} \right]_0^{\pi/2} \right) = \frac{2\pi}{3} R^2 (2-a) \langle u_{EM} \rangle$$

? islatutakoa: $\Delta q = q \cos\theta \rightarrow P_i = r 2 \Delta \vec{q} \cdot \vec{c} = r 2 \langle u_{EM} \rangle \cos^3\theta = (1-a) \cdot 2 \langle u_{EM} \rangle \cos^3\theta$

2.6.) Lurraren orbitan — erradiazio presioa

Totala biena batuz eta integrala $\pi/2$ ra

$\langle \vec{S} \rangle = 1300 \ W/m^2$

a) $P_{err.} = \frac{\langle \vec{S} \rangle}{c} = \frac{1300 \ W/m^2}{3 \cdot 10^8 m/s} = 4'33 \cdot 10^{-6} \ N/m^2$

b) $m = 1000 \ kg$ $\quad a = 2g = 19'6 \ m/s^2$

$F = 2P_{err.} \cdot A = m \cdot a \rightarrow A = \frac{m \cdot a}{2P_{err.}} = 2'26 \cdot 10^9 \ m^2$

$\downarrow$ 2260 $km^2$

(Bizkaiko azalera = 2217 $km^2$)

$\hookrightarrow$ bela islatzaile perfektua bada eragindako presioan bikoitza.

2.7.)

$CO_2 \rightarrow \lambda = 10'6\,\mu m = 10'6 \cdot 10^{-6} m = 1'06 \cdot 10^{-5} m$ (Laser sorta)

$E_0 = 3 \cdot 10^6 V/m$ $\qquad I = \langle \vec{S} \rangle = \frac{1}{2} c \varepsilon_0 \langle E_0^2 \rangle = 12\ GW/m^2$ $\quad (G \equiv 10^9)$

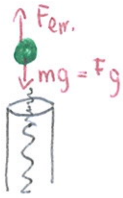
$I$ ↳ irradiantzia

$E_0$ = Eremu elektrikoaren anplitudea

2.8.)

$P = 1000W$ potentziako laserra → argi-sorta lente baten bidez enfokatu material xurgatzaile batera. ($R$ erradioko eta $\rho = 0'25\ g/cm^3$ dentsitateko esfera)


a)




$F_{em} \geq F_g = m \cdot g \longleftrightarrow P_r A = F_{em} = \frac{|\vec{S}|}{c} A = \frac{1}{c} P \geq mg =$ (↗ Presioa; ↗ potentzia → S·A)

$\frac{4}{3}\pi R^3 \rho g \rightarrow R \leq \sqrt[3]{\frac{3P}{4\pi c g \rho}} \simeq 6'87 \cdot 10^{-4}\ m$

b) $T_{urtze} = 800^\circ C = 1073K$

$c_p = 30\ J/KgK$ (bero ahalmena)

$\Delta T \cdot m c_p = P \cdot \Delta t$ → suposatu 0 K-tan zegoela hasieran → $\Delta T = T_{urtze}$

$\Delta T \cdot m c_p$ ⇓ energia termikoa; $P \cdot \Delta t$ ⇓ laserrak emandako energia

$$\Delta t = \frac{\Delta T m c_p}{P} = \frac{T_{urtze}\, m\, c_p}{P} = \frac{T_{urtze} \frac{4\pi}{3} R^3 \rho c_p}{P} = 1'09 \cdot 10^{-5}\ s$$

2.9.)

$$\vec{E} = \frac{E_0 \hat{u}_y}{1 + (x+ct)^2}, \quad \vec{B} = \frac{-B_0 \hat{u}_z}{1+(x+ct)^2}$$

- $$\vec{\nabla} \times \vec{E} = \frac{\partial E}{\partial x} \hat{u}_z = \frac{-E_0 (2(x+ct))}{(1+(x+ct)^2)^2} \hat{u}_z = \frac{-2E_0 (x+ct) \hat{u}_z}{(1+(x+ct)^2)^2} = -\frac{\partial \vec{B}}{\partial t} = -\frac{(+B_0 \hat{u}_z)(2(x+ct) \cdot c)}{(1+(x+ct)^2)^2} =$$

$$-\frac{2B_0 c (x+ct) \hat{u}_z}{(1+(x+ct)^2)^2} \underset{E_0 = B_0 c}{=} -\frac{2E_0 (x+ct)}{(1+(x+ct)^2)^2} \hat{u}_z \quad \checkmark$$

- $$\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

- $$\vec{\nabla} \cdot D \underset{\text{suposatu } D = \varepsilon \vec{E}}{=} \varepsilon\, \vec{\nabla} \cdot \vec{E} = 0 \rightarrow \rho = 0 \text{ (Kargarik ez)}$$

$\cdot\ \vec{\nabla}\times\vec{H} = \frac{1}{\mu}\vec{\nabla}\times\vec{B} = \frac{1}{\mu}\left(-\frac{\partial B}{\partial x}\right)\hat{u}_y = \frac{1}{\mu}\left(\frac{-B_0\cdot 2(x+ct)}{(1+(x+ct)^2)^2}\right)\hat{u}_y = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + \varepsilon\frac{\partial \vec{E}}{\partial t} = \vec{J} + \varepsilon\left(\frac{-E_0 \cdot 2c(x+ct)}{(1+(x+ct)^2)^2}\right)\hat{u}_y =$

Suposatuz $\vec{B} = \mu\vec{H}$

$\vec{J} - \frac{2\varepsilon E_0 c(x+ct)}{(1+(x+ct)^2)^2}\hat{u}_y = \vec{J} - \frac{2\varepsilon B_0 c^2(x+ct)}{(1+(x+ct)^2)^2}\hat{u}_y = \vec{J} - \frac{2\varepsilon B_0 \frac{1}{\varepsilon_0\mu_0}(x+ct)}{(1+(x+ct)^2)^2}\hat{u}_y = \quad \vec{J}=0$

$E_0 = B_0 c$

c abiaduran hedatu. $\varepsilon = \varepsilon_0$ eta $\mu = \mu_0$

$c^2 = 1/\varepsilon_0\mu_0$

$-\frac{2B_0}{\mu_0}\frac{(x+ct)}{(1+(x+ct)^2)^2}\hat{u}_y$ ✓

$\rightarrow \vec{S} = \vec{E}\times\vec{H} = \vec{E}\times\frac{\vec{B}}{\mu_0} = \frac{1}{\mu_0}\left(\frac{E_0\cdot B_0}{(1+(x+ct)^2)^2}\right)(\hat{u}_y)\times(-\hat{u}_z) = \frac{E_0 B_0}{\mu_0(1+(x+ct)^2)^2}(-\hat{u}_x) \Rightarrow$ x-ren

norantza negatiboan hedatzen da.

$\Psi(x+ct)$ funtzioa da → uhin laua → c abiaduraz x-n hedatzen den funtzioa

Gainera $\vec{E}\perp\vec{B}$, $\vec{K}\perp\vec{E},\vec{B}$

$B_0 = \frac{E_0}{c} = 10^{-4}\,T \leftrightarrow E_0 = c\cdot B_0 = 3\cdot 10^8\,m/s\cdot 10^{-4}\,T = 3\cdot 10^4\,V/m$

$\vec{S} = \frac{E_0 B_0}{\mu_0(1+(x+ct)^2)^2}(-\hat{u}_x) = \frac{cB_0^2}{\mu_0(1+(x+ct)^2)^2}(-\hat{u}_x) = S_0\cdot\frac{1}{(1+(x+ct)^2)^2}(-\hat{u}_x)$, $S_0 = cB_0^2 = \cancel{10^{-8}}\,T^2\cdot 3\cdot\cancel{10^8}\,m/s$
$3\,T\cdot V/m =$
$3\,W/m^2$

2.10.)

$\langle\vec{S}\rangle = 1300\,W/m^2$, erradiazioa linealki polarizatutako uhin-laua: $(\theta_1 = \theta_2 \quad \frac{E_x}{E_1} = \frac{E_y}{E_2})$

$\vec{E} = E_1\cos(Kz-\omega t+\theta)\vec{i} + E_2\cos(Kz-\omega t+\theta)\vec{j}$ $\quad |\vec{E}| = \sqrt{E_1^2+E_2^2}\cos(Kz-\omega t+\theta)$

$\vec{B} = \frac{\vec{K}}{\omega}\times\vec{E} \rightarrow \vec{B} = \frac{K}{\omega}[E_1\cos(Kz-\omega t+\theta)\vec{j} - E_2\cos(Kz-\omega t+\theta)\vec{i}] \rightarrow \vec{H} = \frac{\vec{B}}{\mu}$

$1/c$

$\vec{S} = \vec{E}\times\vec{H} = \frac{K}{\omega\mu}(E_1^2\cos^2(Kz-\omega t+\theta)\hat{k} + E_2^2\cos^2(Kz-\omega t+\theta)\hat{k}) \rightarrow \langle\vec{S}\rangle = \frac{K}{\omega\mu}\cdot\frac{1}{2}\underbrace{(E_1^2+E_2^2)}_{E_0^2}$

$|\vec{E}| = \sqrt{\cos^2(Kz-\omega t+\theta)(E_1^2+E_2^2)} = \cos(Kz-\omega t+\theta)\underbrace{\sqrt{E_1^2+E_2^2}}_{E_0} = \cos(Kz-\omega t+\theta)\sqrt{\frac{2\omega\mu}{K}\langle\vec{S}\rangle} = \sqrt{2c\mu_0\langle\vec{S}\rangle}$

$\mu = \mu_0$
$v = c$ (argia)

$(\cos(Kz-\omega t+\theta)) = \cos(Kz-\omega t+\theta)\ 990\,V/m$

$|\vec{B}| = \frac{K}{\omega}\cos(Kz-\omega t+\theta)\sqrt{E_1^2+E_2^2} = \frac{K}{\omega}|\vec{E}| = \frac{1}{c}|\vec{E}| = 3'3\cdot 10^{-6}\,T\cos(Kz-\omega t+\theta)$

2.11.)

"Eskumarantz" zirkularki polarizatutako uhin baten eremu elektrikoa:

$E_1 = E_2 \rightarrow |\theta_1 - \theta_2| = \pi/2 \rightarrow \vec{E} = (E_1 e^{i\theta_1}\vec{i} + E_1 e^{i\theta_2}\vec{j})e^{i(kz-\omega t)} =$ $\nearrow$ $\theta_1 - \theta_2 = \pi/2$

$(E_1 e^{i\theta_1}\vec{i} + E_1 e^{i(\theta_1 - \pi/2)}\vec{j})e^{i(kz-\omega t)} = E_1 e^{i\theta_1}(\vec{i} + e^{-i\pi/2}\vec{j})e^{i(kz-\omega t)} =$

$E_1(\vec{i} - i\vec{j})e^{i(kz-\omega t+\theta_1)}$ (Ezkerrerantz: $\vec{E} = E_1(\vec{i} + i\vec{j})e^{i(kz-\omega t+\theta_1)}$)

$\hookrightarrow |\theta_1 - \theta_2| = \theta_2 - \theta_1 = \pi/2 \rightarrow \theta_2 = \theta_1 + \pi/2$

2.12.)

Eskumarantz: $\vec{E}_+ = E_0(\vec{i} - i\vec{j})e^{i(kz-\omega t+\varphi)}$ ; Ezkerrerantz $\vec{E}_- = E_0(\vec{i} + i\vec{j})e^{i(kz-\omega t+\varphi)}$

$\vec{E} = \vec{E}_+ + \vec{E}_- = 2E_0\vec{i}\, e^{i(kz-\omega t+\varphi)}$ $\Rightarrow$ linealki polarizatuta x norabidean

2.13.)

$f = f_0 e^{i(\omega t+\varphi)}$ $g = g_0 e^{i(\omega t+\varphi)}$ $\longrightarrow \langle Re(f) Re(g)\rangle = \frac{1}{2} Re(f \cdot g^*)$

$\vec{S} = \vec{E} \times \vec{H} = Re(\vec{E}_c) \times Re(\vec{H}_c) \rightarrow \langle|\vec{S}|\rangle = \langle Re(\vec{E}_c) Re(\vec{H}_c)\rangle = \frac{1}{2} Re(E_c \cdot H_c^*) = \frac{1}{2} Re(\vec{E}_0 \vec{H}_0) = I$

$\left(\begin{array}{l} \vec{E}_c = \vec{E}_0 e^{-i\omega t} \\ \vec{H}_c = \vec{H}_0 e^{-i\omega t} \end{array}\right)$

$u = \frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu}B^2 = \frac{1}{2}\varepsilon (Re(\vec{E}_c))^2 + \frac{1}{2\mu}(Re(\vec{B}))^2 \rightarrow \langle u\rangle = \frac{1}{2}\varepsilon\langle (Re(\vec{E}_c))^2\rangle + \frac{1}{2\mu}\langle (Re(\vec{B}))^2\rangle =$

$\frac{1}{2}\varepsilon \cdot \frac{1}{2} Re(E_c \cdot E_c^*) + \frac{1}{2\mu} \cdot \frac{1}{2} Re(B_c \cdot B_c^*) = \frac{1}{4}\varepsilon Re(E_0^2) + \frac{1}{4\mu} Re(B_0^2) = \frac{1}{4}(\varepsilon Re(E_0^2) + \mu Re(H_0^2))$

$\downarrow$

$\vec{B} = \mu\vec{H}$

2.14.)

$\varepsilon^* = \varepsilon' + i\varepsilon''$ ; $\mu^* = \mu' + i\mu''$ ; $\sigma^* = \sigma' + i\sigma''$

(Teorian 4. puntuan)

2.15)

$\sigma' = 4\ (\Omega m)^{-1} = 4\ S/m$ , $\varepsilon = \varepsilon_0$ $\mu = \mu_0$ $Q = \frac{\omega\varepsilon}{\sigma}$

a). $\nu = 10^2\ Hz$ (potentzia elektrikoa) $\rightarrow$ $Q = \frac{2\pi\nu\varepsilon_0}{\sigma} = 1'39 \cdot 10^{-9} << 1$ eroale oso ona

* $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{2}{\sigma\mu 2\pi\nu}} = 25'16\ m \rightarrow \beta = \frac{1}{\delta} = 3'97 \cdot 10^{-2}\ m^{-1} = \alpha$

* $\Omega = \text{arctg}\ \frac{\beta}{\alpha} = \text{arctg}\ 1 = \frac{\pi}{4}$ (E eta B-ren desfase erlatiboa)

* $\frac{|\vec{B}_{real}|}{|\vec{E}_{real}|} = \sqrt{\mu_0\varepsilon_0} = \frac{1}{c}$

b). $\nu = 10^7\ Hz$ (irratia) $\rightarrow$ $Q = \frac{\omega\varepsilon}{\sigma} = \frac{2\pi\nu\varepsilon_0}{\sigma} = 1'39 \cdot 10^{-4}$ eroale ona.

$\alpha = \left(\frac{1}{2}\mu\sigma 2\pi\nu\right)^{1/2}\left(1 + \frac{1}{2}Q\right) = 12'56\ m^{-1}$ $\beta = \left(\frac{1}{2}\mu\sigma 2\pi\nu\right)^{1/2}\left(1 - \frac{1}{2}Q\right) = 12'56\ m^{-1}$

* $\delta = \frac{1}{\beta} = 7'95 \cdot 10^{-2}\ m$ * $\Omega = \text{arctg}\ \frac{\beta}{\alpha} \simeq \frac{\pi}{4}$ * $\frac{|B|}{|E|} = \sqrt{\frac{\mu\sigma}{2\pi\nu}} = 2'83 \cdot 10^{-7}\ s/m$

c). $\nu = 10^{10}\ Hz$ (mikrouhinak) $\rightarrow$ $Q = \frac{\omega\varepsilon}{\sigma} = 0'139$ $\beta = 2\pi\nu\sqrt{\frac{\mu_0\varepsilon_0}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{2\pi\nu\varepsilon_0}\right)^2} - 1\right]^{1/2} = 3'56 \cdot 10^{2}\ m^{-1}$

* $\delta = \frac{1}{\beta} = 2'8 \cdot 10^{-3}\ m$ $\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{2\pi\nu\varepsilon_0}\right)^2} + 1\right]^{1/2} = 4'26 \cdot 10^{2}\ m^{-1}$

* $\Omega = \text{arctg}\ \frac{\beta}{\alpha} = 0'696\ rad \simeq 39'89°$ * $\frac{|B_{real}|}{|G_{real}|} = \sqrt{\mu_0\varepsilon_0}\left(1 + \frac{1}{Q^2}\right)^{1/4} = 8'98 \cdot 10^{-9}\ s/m$

d). $\nu = 10^{15}\ Hz$ (argia) $\rightarrow$ $Q = \frac{\omega\varepsilon}{\sigma} = \frac{2\pi\nu\varepsilon_0}{\sigma} = 1'39 \cdot 10^{4} >> 1$ dielektriko ona.

dielektrikoen delako

$\alpha = \omega\sqrt{\mu\varepsilon_0\varepsilon_r}\left(1 + \frac{1}{8Q^2}\right) = 1'879 \cdot 10^{8}\ m^{-1}$ $\beta = \frac{\sigma}{2}\left(\frac{\mu}{\varepsilon_0\varepsilon_r}\right)^{1/2} = 83'99\ m^{-1}$

* $\delta = \frac{1}{\beta} = 1'19 \cdot 10^{-2}\ m$ * $\Omega = \text{arctg}\left(\frac{\sigma}{2\omega\varepsilon}\right) = 4'46 \cdot 10^{-7}\ rad$

$\hookrightarrow \text{arctg}\ \frac{\beta}{\alpha}$

* $\frac{|B|}{|E|} = \sqrt{\mu_0\varepsilon_0\varepsilon_r}\left(1 + \frac{1}{4Q^2}\right) = 2'99 \cdot 10^{-8}\ s/m$

Oso portaera ezberdina $\nu$-ren arabera.

2.16)

Irrati uhina $\rightarrow$ uhin laua ; ionosferan hedatu da kargadun partikula askorekin elkar eragin

Partikula horiek uhinaren eremu magnetikoarekiko perpendikularki eta 0'1c abiaduran higitzen dira. $(v = 0'1c)$

$\frac{F_e}{F_m}$? (Uhinak partikulen gainean eragiten duen indar elektrikoaren eta magnetikoaren arteko zatidura)

$\psi = \psi_0 e^{i(kz - \omega t + \varphi_0)}$ ; $\vec{E} = \vec{E}_0 e^{i(kz - \omega t + \varphi_0)}$ ; $\vec{B} = \vec{B}_0 e^{i(kz - \omega t + \varphi_0)}$ $(\vec{E} \perp \vec{B})$

$F_e = qE$ (q partikulen karga izanik) ; $F_m = q(\vec{v} \times \vec{B}) \underset{\vec{v} \perp \vec{B}}{=} qvB = q\frac{c}{10}B$

$$\frac{F_e}{F_m} = \frac{qE}{q\frac{c}{10}B} = \frac{10E}{cB} = \frac{10E_0}{cB_0} \underset{B_0 = \frac{E_0}{v}}{=} \frac{10E_0}{c\frac{E_0}{v}} = 10\frac{v}{c} \underset{v=c}{=} 10$$

2.17.)

Uhin lauak: $\vec{E} = \vec{E}_0 e^{i(kz - \omega t + \varphi_0)}$ $\vec{H} = \vec{H}_0 e^{i(kz - \omega t + \varphi_0)}$

$$\frac{|\vec{E}|}{|\vec{H}|} = \eta = \frac{E_0}{H_0} \rightarrow [\eta] = \frac{V/m}{A/m} = \Omega$$

Dielektriko perfektua: $\eta = \frac{E_0}{H_0} = \frac{E_0}{\frac{1}{v\mu}E_0} = v\mu \underset{v = \frac{1}{\sqrt{\varepsilon\mu}}}{=} \frac{\mu}{\sqrt{\varepsilon\mu}} = \sqrt{\frac{\mu}{\varepsilon}}$

Hutsaren inpedantzia: $\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega$

2.18.)

Eroale baten berezko inpedantzia: $\eta = a + bi$ $a, b \in \mathbb{R}$

Eroale ona $(Q \ll 1)$: $\eta = \frac{|\vec{E}|}{|\vec{H}|} = \frac{|\vec{E}|}{|\vec{B}|}\frac{\mu}{e^{i\Omega}} = \mu \cdot \frac{1}{\frac{|\vec{B}|}{|\vec{E}|}} = \frac{\mu}{\left(\frac{\mu\varepsilon}{Q}\right)^{1/2}} = \frac{\mu\sqrt{Q}}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu Q}{\varepsilon}}$

fase falta da.

Desfasea: $\Omega = \arctan(1-Q) \simeq \pi/4$

2.19.)

Indargatz-koefizientea $\rightarrow$ dB/m

$$\frac{1\,dB}{m} = 10\log\frac{I_1}{I_i}$$ → bi punturen arteko intentsitateen arteko zatidura
↓ 1 m-tara

* Galera gutxiko materiala: eroale txarra:

$$\vec{E} = \vec{E}_{0a}\, e^{-\beta z}\cos(\alpha z - \omega t + \theta) \qquad \vec{B} = \frac{|K|}{\omega}\vec{E}_{0a}\, e^{-\beta z}\cos(\alpha z - \omega t + \theta + \Omega)$$

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu}\frac{|K|}{\omega}E_{0a}^2 e^{-2\beta z}\cos(\alpha z - \omega t + \theta)\cos(\alpha z - \omega t + \theta + \Omega) = \frac{1}{2\mu}\frac{|K|}{\omega}E_{0a}^2 e^{-2\beta z}\left(\cos(\Omega) + \cos\left(2\alpha z - 2\omega t + 2\theta + \Omega\right)\right)$$

$$I = \langle\vec{S}\rangle = \frac{1}{2\mu}\frac{|K|}{\omega}E_{0a}^2\cos\Omega\, e^{-2\beta z}$$

$$K = 10\log\frac{I(z = z_0 + 1)}{I(z = z_0)} = 10\cdot\log\frac{\cancel{e^{-2\beta z_0}}\cdot e^{-2\beta}}{\cancel{e^{-2\beta z_0}}} = 10\log e^{-2\beta} \qquad \beta = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

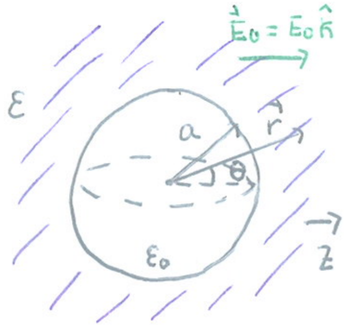
* Galera handiko materiala: eroale ona:

$$K = 10\log e^{-2\beta} \qquad \beta = \sqrt{\frac{\mu\sigma\omega}{2}}\left(1 - \frac{1}{2}Q\right)$$

# 1. GAIA: ETXERAKO ARIKETAK

## 11. taldea

### 1. Etxerako ariketa:




a erradioko burbuila ($\varepsilon_0$ permitibitateduna) esferikoa hasieran uniformea den $\vec{E}_0 = E_0 \hat{k}$ eremu elektrikoan kokatzen da.

Burbuilaren barneko eremua kalkulatzeko Laplace-n ekuazioa ebatziko dugu; lehendabizi:

Hauek dira ezagutzen ditugun mugalde baldintzak:

1. $\vec{E}(r,\theta)\big|_{r\to\infty} = \vec{E}_0 = E_0\hat{k} = -\vec{\nabla}\phi \rightarrow \phi(r,\theta)\big|_{r\to\infty} = -E_0 r\cos\theta$

2. $\phi(r,\theta)$ potentziala jarraitua da $r=a$ puntuan.

3. Gainazalean ez dago karga askerik $\rightarrow D_{1r} = D_{2r}\big|_{r=a}$

   ↳ erradialak


4. $\phi$ finitua izan behar da jatorrian

Halaber, simetriagatik badakigu potentziala eta eremua $\varphi$ angeluaren independenteak izango direla: $\phi = \phi(r,\theta)$ eta $\vec{E} = \vec{E}(r,\theta)$. Horregatik Laplacen ekuazioa ebatziz honako itxura izango du potentzialak:

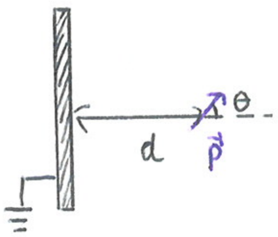
$$\nabla^2\phi = 0 \rightarrow \phi(r,\theta) = \sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+1)}\right] P_n(\cos\theta)$$

Potentziala kalkulatuko dugu alde batetik burbuilaren barnean ($\phi_1$) eta bestetik kanpoaldean ($\phi_2$) eta mugalde baldintzak aplikatu:

1 ⇒ $r < a$ : $\nabla^2\phi = 0 \rightarrow \phi_1(r,\theta) = \sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+1)}\right] P_n(\cos\theta)$

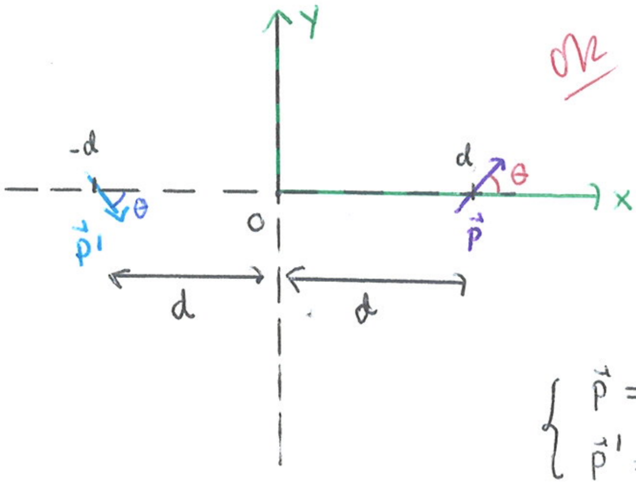
• $\phi$ finitua dela dakigunez jatorrian, $B_n$-ak nuluak izan behar dira

## 3. Etxerako ariketa:




Xafla eroale infinitua lurrarekin lotuta eta d distantziara p momentuko dipoloa dauzkagunez problema hau ebazteko karga induzituaren metodoa aplikatuko dugu.

Hauxe izango da gure problema baliokidea:




Hau da, koordenatu sistemaren jatorria plano eroalean kokatuz (-d,0) puntuan dipolo induzitua kokatu dugu, $\vec{p}'$ momentu dipolarrekoa:

$$\begin{cases} \vec{p} = p(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) \\ \vec{p}' = p(\cos\theta\,\vec{i} - \sin\theta\,\vec{j}) \end{cases}$$

Izan ere, horrela gure jatorrizko problemaren mugalde baldintza betetzen da, $\phi(x=0)=0$ (eroalea lurrera lotuta baitago):

- Dipolo batek sortutako potentziala $\phi(\vec{r}) = \dfrac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$ bada $\Longrightarrow$

$$\phi(0,y) = \phi_{\vec{p}'}(0,y) + \phi_{\vec{p}}(0,y) = \frac{\vec{p}'\cdot(d\vec{i}+y\vec{j})}{4\pi\varepsilon_0\sqrt{d^2+y^2}} + \frac{\vec{p}\cdot(-d\vec{i}+y\vec{j})}{4\pi\varepsilon_0\sqrt{d^2+y^2}} =$$

$$\frac{1}{4\pi\varepsilon_0}\cdot\frac{p}{\sqrt{d^2+y^2}}(\cancel{\cos\theta\cdot d} + \cancel{\sin\theta\cdot y} - \cancel{\cos\theta\cdot d} - \cancel{\sin\theta\cdot y}) = 0$$

Horrela, problema baliokidean xafla eta dipoloaren arteko indarra kalkulatzeko $\vec{p}'$ dipolo induzituak $\vec{p}$ dipoloaren gainean eragiten duen indarra kalkulatuko dugu. Horretarako lehendabizi $\vec{p}'$ dipoloak gure koordenatu sistemako X ardatzeko puntuetan sortzen duen eremua kalkulatuko dugu. $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0 r^3}\left(\dfrac{3(\vec{p}'\cdot\vec{r})\vec{r}}{r^2} - \vec{p}'\right)$ dipolo batek sortzen duen eremua bada $\vec{r}$ dipolotik puntura doan bektorea izanda, hauxe

da $\vec{p}\,'$ dipoloak sortuko duen eremua X ardatzeko puntuetan; hau da

$\vec{r} = (x+d)\vec{i}$ izanda:

$$* \quad \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\left(\frac{3(\vec{p}\,'\cdot\vec{r})\cdot\vec{r}}{r^2} - \vec{p}\,'\right) \Rightarrow \vec{E}(x) = \frac{1}{4\pi\varepsilon_0(x+d)^3}\left(\frac{3(\vec{p}\,'\cdot\cancel{(x+d)}\vec{i})\cdot\cancel{(x+d)}\vec{i}}{\cancel{(x+d)^2}} - \vec{p}\,'\right) =$$

$$\frac{1}{4\pi\varepsilon_0(x+d)^3}\left[\left[3p(\cos\theta\,\vec{i} - \sin\theta\,\vec{j})\cdot\vec{i}\right]\vec{i} - p(\cos\theta\,\vec{i} - \sin\theta\,\vec{j})\right] = \frac{p}{4\pi\varepsilon_0(x+d)^3}\left(2\cos\theta\,\vec{i} + \sin\theta\,\vec{j}\right)$$

Halaber, $\vec{F} = -\vec{\nabla}U$ izanda eta $U = -\vec{p}\cdot\vec{E}$ $\vec{p}$ momentu dipolarreko

dipolo elektrikoak $\vec{E}$ kanpo eremu elektrikoaren menpean duen energia

potentziala, X ardatzean kokatutako dipoloak jasango duen indarra

hauxe izango da:

$$\circ \quad \vec{F} = -\vec{\nabla}U = -\vec{\nabla}(-\vec{p}\cdot\vec{E}) = \vec{\nabla}(\vec{p}\cdot\vec{E}) = \vec{\nabla}\left(p(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})\cdot\frac{p}{4\pi\varepsilon_0}\frac{(2\cos\theta\,\vec{i} + \sin\theta\,\vec{j})}{(x+d)^3}\right) =$$

$$\frac{p^2}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{2\cos^2\theta + \sin^2\theta}{(x+d)^3}\right) = \frac{p^2}{4\pi\varepsilon_0}(2\cos^2\theta + \sin^2\theta)\vec{\nabla}\left(\frac{1}{(x+d)^3}\right) = \frac{p^2}{4\pi\varepsilon_0}(1+\cos^2\theta)\frac{d}{dx}\left(\frac{1}{(x+d)^3}\right)\vec{i} =$$

$$-\frac{3p^2(1+\cos^2\theta)}{4\pi\varepsilon_0(x+d)^4}\vec{i}$$

Beraz, $x = d$ puntuan kokatuta dagoen gure $\vec{p}$ dipoloak jasango duen

indarra $\boxed{\vec{F}(d) = -\frac{3p^2(1+\cos^2\theta)}{64\pi\varepsilon_0 d^4}\vec{i}}$ izango da, plano eroaleak eragingo

beti erakarle!!!

dionaren berdina.

Gainera maximoa izango da indar hori honako $\theta$ angeluarekin:

$$|\vec{F}| = \frac{3p^2}{64\pi\varepsilon_0 d^4}(1+\cos^2\theta)$$

$$\frac{d|\vec{F}|}{d\theta} = -\frac{3p^2}{64\pi\varepsilon_0 d^4}(2\cos\theta\sin\theta) = 0 \quad \longleftrightarrow \quad \cos\theta = 0 \quad \text{edo} \quad \sin\theta = 0$$

$\Rightarrow \cos\theta = 0 \quad \longleftrightarrow \quad \theta = \frac{\pi}{2} \; ; \; \frac{3\pi}{2}$ .

$$\theta = \frac{\pi}{2} \rightarrow \quad \frac{d^2|\vec{F}|}{d\theta^2} = -\frac{3p^2}{64\pi\varepsilon_0 d^4}(-2\sin^2\theta + 2\cos^2\theta) = -\frac{3p^2}{32\pi\varepsilon_0 d^4}(\cos^2\theta - \sin^2\theta)$$

$$\frac{d^2|\vec{F}|}{d\theta^2}\left(\frac{\pi}{2}\right) = \frac{+3p^2}{32\pi\varepsilon_0 d^4} > 0 \rightarrow \text{~~Maximoa~~} \quad \text{Minimoa}$$

$$\theta = \frac{3\pi}{2} \rightarrow \quad \frac{d^2|\vec{F}|}{d\theta^2}\left(\frac{3\pi}{2}\right) = \frac{3p^2}{32\pi\varepsilon_0 d^4} > 0 \rightarrow \text{~~Maximoa~~} \quad \text{Minimoa}$$

$\Rightarrow \sin\theta = 0 \quad \longleftrightarrow \quad \theta = 0, \pi$

$$\theta = 0 \rightarrow \quad \frac{d^2|\vec{F}|}{d\theta^2}(0) = \frac{-3p^2}{32\pi\varepsilon_0 d^4} < 0 \rightarrow \quad \text{Maximoa}$$

$$\theta = \pi \rightarrow \quad \frac{d^2|\vec{F}|}{d\theta^2}(\pi) = \frac{-3p^2}{32\pi\varepsilon_0 d^4} < 0 \rightarrow \quad \text{Maximoa}$$

Beraz indarra maximoa izango da $\theta = 0, \pi$ kasuetarako (moduluz) eta kasu horietan honixe izango da dipoloak jasango duen indarra:

$$\theta = 0 \rightarrow \quad \vec{F} = \frac{-3p^2}{32\pi\varepsilon_0 d^4}\,\vec{i} \quad , \quad \theta = \pi \rightarrow \quad \vec{F} = \frac{-3p^2}{32\pi\varepsilon_0 d^4}\,\vec{i}$$

3. Etxerako ariketa:

1) Maxwell-en ekuazioen bidez, froga ezazu eremu elektriko eta magnetikoak uhin-funtzioa betetzen dutela. Eta adierazi zein abiadurarekin hedatzen diren uhin horiek.

Hutsean: $\rho = 0$ $\vec{J} = 0$

Maxwell-en ekuazioak:

OK

(1) $\nabla \cdot \vec{E} = 0$ (2) $\nabla \cdot \vec{B} = 0$

(3) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (4) $\nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$

$$\frac{\partial}{\partial t}\left[\nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right] \Rightarrow \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla \times \frac{\partial \vec{B}}{\partial t} \overset{(3)}{=} -\nabla \times (\nabla \times \vec{E})$$

$$-\nabla \times (\nabla \times \vec{E}) = -[\nabla(\underbrace{\nabla \cdot \vec{E}}_{0\ (1)}) - \nabla^2 \vec{E}] = \nabla^2 \vec{E}$$

$$\boxed{\nabla^2 \vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

$$\frac{\partial}{\partial t}\left[\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\right] \Rightarrow -\frac{\partial^2 \vec{B}}{\partial t^2} = \nabla \times \frac{\partial \vec{E}}{\partial t} \overset{(4)}{=} \frac{1}{\varepsilon_0 \mu_0} \nabla \times (\nabla \times \vec{B})$$

$$\frac{1}{\varepsilon_0 \mu_0} \nabla \times (\nabla \times \vec{B}) = \frac{1}{\varepsilon_0 \mu_0}[\nabla(\underbrace{\nabla \cdot \vec{B}}_{0\ (2)}) - \nabla^2 \vec{B}] = -\frac{\nabla^2 \vec{B}}{\varepsilon_0 \mu_0}$$

$$\boxed{\nabla^2 \vec{B} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$

Uhin funtzioarekin konparatuz: $\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$

Ondorioztatzen dugu $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ dela uhin elektromagnetikoen abiadura.

2) Zergatik karga irudikarioen metodoaz lortutako emaitza jatorrizko problemaren bardina da konfigurazio guztiz desberdinak badituzte?

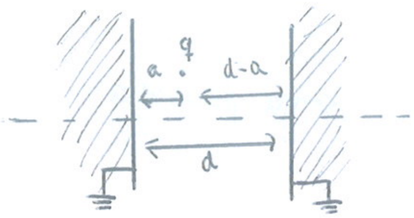
karga irudikarioen metodoan karga irudikariak jatorrizko problemaren mugalde baldintzak betetzeko kokatzen direlako eta soluzioaren bakartasunaren teoremak ziurtatzen duelako muga-baldintza jakin batzuetarako Laplace-n edo Poisson-en ebazpena; hau da, atomizko problemaren ebazpena, bakarra dela.

# ELEKTROMAGNETISMOA II

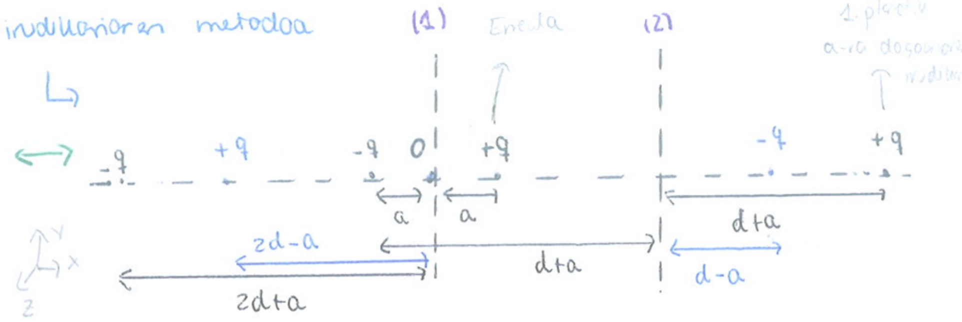
16-09-26

## KARGA IRUDIKARIEN METODOA:

1.15.)




* Karga-irudikariaren metodoa




* Horrela, infinitu karga irudikari behar ditugu:
  - 1. planotik ezkerrera:
    - $-q$: a distantziara, $2d+a$ distantziara, $4d+a$ distantziara.... ( $2nd+a$ $n\in\mathbb{N}$)
    - $+q$: $2d-a$ distantziara, $4d-a$ distantziara,... ( $2(n+1)d-a$ $n\in\mathbb{N}$)
  - 2. planotik eskuinera:
    - $-q$: $d-a$ distantziara, $3d-a$ distantziara,... ( $(2n+1)d-a$ $n\in\mathbb{N}$ )
    - $+q$: $d+a$ distantziara, $3d+a$ distantziara,... ( $(2n+1)d+a$ $n\in\mathbb{N}$ )

* Planoetan induzitutako karga:

1 planoan

$$1 \Rightarrow (\vec{E}_2-\vec{E}_1)\Big|_{x=0}\cdot \underset{\hat{i}}{\hat{n}} = \frac{\sigma_1}{\varepsilon_0} = +E_x\Big|_{x=0} \to \sigma_1 = +\varepsilon_0 E_x\Big|_{x=0} \overset{*}{=} +\frac{\varepsilon_0 q}{4\pi\varepsilon_0}\left(\frac{-a}{(a^2+R^2)^{3/2}}+\sum_{n=0}^{\infty}\left[\frac{2(n+1)d+a}{((2(n+1)d+a)^2+R^2)^{3/2}}-\frac{(a+2nd)}{((2nd+a)^2+R^2)^{3/2}}\right]\right)$$

*

- $$\phi(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-a)^2+y^2+z^2}}+\sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{(x+2(n+1)d-a)^2+y^2+z^2}}+\frac{1}{\sqrt{(x-(2(n+1)d-a))^2+y^2+z^2}}-\frac{1}{\sqrt{(x-(2(n+1)d+a))^2+y^2+z^2}}-\frac{1}{\sqrt{(x+2nd+a)^2+y^2+z^2}}\right]\right)$$

- $$E_x=-\frac{\partial\phi}{\partial x}=\frac{q}{4\pi\varepsilon_0}\left(\frac{x-a}{((x-a)^2+y^2+z^2)^{3/2}}+\sum_{n=0}^{\infty}\left[\frac{(x-a+2(n+1)d)}{((x+2(n+1)d-a)^2+y^2+z^2)^{3/2}}+\frac{(x+a-2(n+1)d)}{((x-(2(n+1)d-a))^2+y^2+z^2)^{3/2}}-\frac{(x+a+2dn)}{((x+2nd+a)^2+y^2+z^2)^{3/2}}-\frac{(x-2(n+1)d-a)}{((x-(2(n+1)d+a))^2+y^2+z^2)^{3/2}}\right]\right)$$

$$\to E_x\Big|_{x=0}=\frac{q}{4\pi\varepsilon_0}\left(\frac{-a}{(a^2+R^2)^{3/2}}+\sum_{n=0}^{\infty}-\frac{(a+2dn)}{((2nd+a)^2+R^2)^{3/2}}+\frac{2(n+1)d+a}{((2(n+1)d+a)^2+R^2)^{3/2}}\right)$$

$$y^2+z^2=R^2$$

$$Q_1 = \int_0^\infty \sigma_1 \cdot 2\pi R\, dR = -\frac{q}{2}\left[\int_0^\infty \frac{aR\,dR}{(a^2+R^2)^{3/2}} + \sum_{n=0}^{\infty}\left[\int_0^\infty\left(\frac{(a+2nd)R}{((2nd+a)^2+R^2)^{3/2}} - \frac{(2(n+1)d+a)R}{((2(n+1)d+a)^2+R^2)^{3/2}}\right)dR\right]\right] =$$

$$-\frac{q}{2}\left(1+\sum_{n=0}^{\infty}\left[-\frac{(2(n+1)d+a)}{\sqrt{((a+2(n+1)d)^2+R^2)}} + \frac{(a+2nd)}{\sqrt{((a+2nd)^2+R^2)}}\right]_0^\infty\right) = -\frac{q}{2}\left(1+\sum_{n=0}^{\infty}\cancel{(1-1)}\right) = -\frac{q}{2}$$

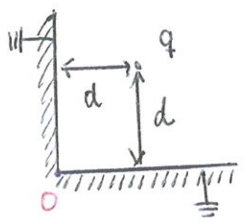
? * Gauza $-1+1-1+1-1+\dots = -1/2$

Gauza bera 2. plano eroalean, $Q_2 = -\frac{q}{2}$

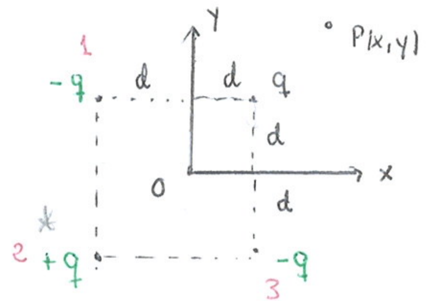
Ez du garrantzirik karga puntualaren posizioak

## 1.17.)

Bi xafla eroale infinitu → $d$ distantziara $q$ baliodun karga puntuala



Karga induzituen metodoa ⟷ problema baliokidea



* Hau jarri beste aldean karga netoa $-q$ izan dadin → karga eroalean distribuzioa

$$\underset{x,y\geq 0}{\phi(x,y)} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-d)^2+(y-d)^2}} - \underbrace{\frac{1}{\sqrt{(x+d)^2+(y-d)^2}}}_{1} + \underbrace{\frac{1}{\sqrt{(x+d)^2+(y+d)^2}}}_{2} - \underbrace{\frac{1}{\sqrt{(x-d)^2+(y+d)^2}}}_{3}\right]$$

Kargaren gaineko indarra: $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \frac{1}{4\pi\varepsilon_0} q^2 \left[\frac{-\hat{\imath}}{4d^2} - \frac{\hat{\jmath}}{4d^2} + \frac{(\hat{\imath}+\hat{\jmath})}{8d^2\sqrt{2}}\right] =$

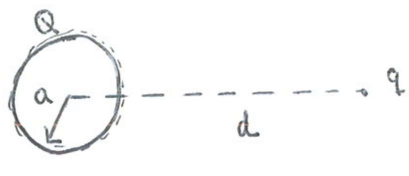
$$\frac{q^2}{16\pi\varepsilon_0 d^2}\left[-\hat{\imath}-\hat{\jmath}+\frac{(\hat{\imath}+\hat{\jmath})}{2\sqrt{2}}\right] = \frac{q^2}{16\pi\varepsilon_0 d^2}(-0'6464(\hat{\imath}+\hat{\jmath})) = -\frac{q^2}{4\pi\varepsilon_0 d^2}\cdot 0'1616(\hat{\imath}+\hat{\jmath})$$

(erakarleia)

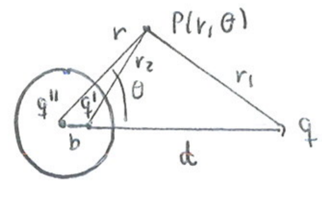
## 1.18.) (entregatzekoa)

## 1.19.)

* Lehenengo $q'$ jarri $b$ distantziara $V(r=R,\theta)=0$ izan dadin eta gero jarri $q''$ esferan potentziala 0 izan ez dadin



problema baliokidea ⟷ (superposizio baliokidea) *



Aurreko problematik badakigu $b = a^2/d$ eta $q' = -\frac{a}{d}q$

Superposizioa: [esfera $\phi = 0$ potentzialera + karga puntuala] + $q''$ karga induzitua zentroan $\phi(r=R) \neq 0$ izan dadin

$P(r,\theta)$ edozein puntuton, $r_1=\sqrt{r^2+d^2-2rd\cos\theta}$ $\quad r_2=\sqrt{r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta}$ izanda →

$$* \quad \phi(r,\theta)=\phi_q(r,\theta)+\phi_{q'}(r,\theta)+\phi_{q''}(r,\theta)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1}+\frac{q'}{r_2}+\frac{q''}{r}\right)=$$

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}}-\frac{aq/d}{\sqrt{r^2+\frac{a^4}{d^2}-2\frac{a^2}{d}r\cos\theta}}+\frac{q''}{r}\right)$$

→ hemen ez dago potentziala zehaztuta, karga baizik

* Esfera isolatua denez $Q=q'+q''$ izango da, $q''=Q-q'=Q+\frac{a}{d}q$

↘ esferaren karga beti da Q eta induzitutako karga karga induzitzen baitu dago $Q=q'+q''$

* Esferaren potentziala $\phi(a,\theta)=\frac{1}{4\pi\varepsilon_0}\left(\cancel{\frac{q}{\sqrt{a^2+d^2-2ad\cos\theta}}}\;\cancel{\frac{-aq/d}{\sqrt{a^2+\frac{a^4}{d^2}-2\frac{a^3}{d}\cos\theta}}}+\frac{Q+\frac{a}{d}q}{a}\right)=$ (zero baitu da)

$$\frac{1}{4\pi\varepsilon_0}\frac{\left(Q+\frac{a}{d}q\right)}{a}=\frac{q''}{4\pi\varepsilon_0 a}=\phi_0$$

$$\sigma(a,\theta)=-\varepsilon_0\frac{\partial\phi}{\partial r}\Big|_{r=a}=-\varepsilon_0\cdot\frac{1}{4\pi\varepsilon_0}\left(\frac{-q(r-d\cos\theta)}{(r^2+d^2-2rd\cos\theta)^{3/2}}+\frac{aq}{d}\frac{\left(r-\frac{a^2}{d}\cos\theta\right)}{\left(r^2+\frac{a^4}{d^2}-2\frac{a^2}{d}r\cos\theta\right)^{3/2}}-\frac{\left(Q+\frac{a}{d}q\right)}{r^2}\right)\Bigg|_{r=a}=$$

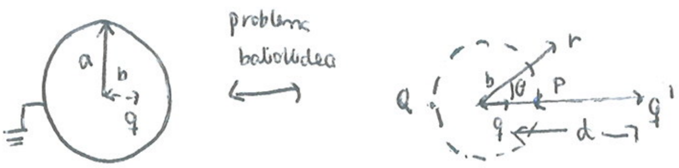
$$-\cancel{\varepsilon_0}\cdot\frac{1}{4\pi\cancel{\varepsilon_0}}\left(\frac{-q(a-d\cos\theta)}{\left(\sqrt{a^2+d^2-2ad\cos\theta}\right)^3}+\frac{a\cancel{q}}{\cancel{d}}\frac{\left(a-\frac{a^2}{d}\cos\theta\right)}{\left(\sqrt{a^2+\frac{a^4}{d^2}-2\frac{a^3}{d}\cos\theta}\right)^3}-\frac{Q+\frac{a}{d}q}{a^2}\right)=$$

$$\frac{1}{4\pi}\left(\frac{-q(d^2-a^2)}{a(a^2+d^2-2ad\cos\theta)^{3/2}}+\frac{Q+\frac{a}{d}q}{a^2}\right)$$

* edo $\phi_0 \overset{\text{esferaren potentziala}}{=} \frac{Q}{4\pi\varepsilon_0 a}+\frac{q}{4\pi\varepsilon_0 d}=\frac{q''}{4\pi\varepsilon_0 a}\rightarrow q''=Q+q\frac{a}{d}$

1.20)

Esfera eroale hutsa → b distantziara q karga




Mugalde baldintza → $\phi(a,\theta)=0$

- $P\rightarrow\phi(a,0)=\frac{q'}{4\pi\varepsilon_0(d-a)}+\frac{q}{4\pi\varepsilon_0(a-b)}=0\rightarrow$

$$q'(a-b)=-q(d-a)\rightarrow q'=-q\frac{(d-a)}{(a-b)}$$

- $Q\rightarrow\phi(a,\pi)=\frac{q'}{4\pi\varepsilon_0(d+a)}+\frac{q}{4\pi\varepsilon_0(b+a)}=0\rightarrow\quad q'=-q\frac{(d+a)}{(b+a)}$

$$\Big\}\; -q\frac{(d-a)}{(a-b)}=-q\frac{(d+a)}{(b+a)}\rightarrow$$

$$(d-a)(b+a)=db+\cancel{da}-\cancel{ab}-\cancel{a^2}=(d+a)(a-b)=\cancel{da}-db+a^2-\cancel{ab}\rightarrow 2db=2a^2\rightarrow d=\frac{a^2}{b}$$

$$q'=-q\frac{\left(\frac{a^2}{b}-a\right)}{(a-b)}=-q\frac{(a^2-ab)}{b(a-b)}=-q\frac{a(a-b)}{b(a-b)}=-q\frac{a}{b}$$

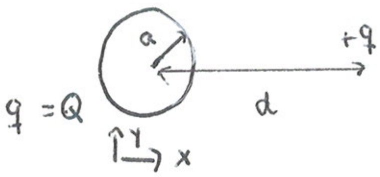
Orain potentziala baliokide da baliogarria esferaren barnean:

$$\phi(r,\theta) = \frac{q}{4\pi\varepsilon_0\sqrt{r^2+b^2-2rb\cos\theta}} + \frac{q'}{4\pi\varepsilon_0\sqrt{r^2+d^2-2rd\cos\theta}} \qquad r<a$$

Zentruan $\rightarrow$ $\phi(a,\theta) = \dfrac{q}{4\pi\varepsilon_0 b} + \dfrac{-q\frac{a}{b}}{4\pi\varepsilon_0 d} = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{b} - \dfrac{a}{bd}\right) = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{b} - \dfrac{\cancel{a}}{\cancel{b}\frac{a^2}{\cancel{b}}}\right) =$

$$\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$$

1.21.)




Problema baliokidean orain: (Esfera kargatuta dagoenean $Q = q$)

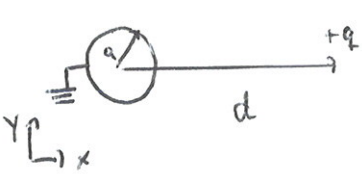
$$\vec{F} = \frac{q'q}{4\pi\varepsilon_0(d-b)^2}\vec{i} + \frac{q''q}{4\pi\varepsilon_0 d^2}\vec{i} = \frac{q}{4\pi\varepsilon_0}\vec{i}\left(\frac{q'}{(d-b)^2} + \frac{q''}{d^2}\right) \underset{q''=Q-q'}{=}$$

$$\frac{q}{4\pi\varepsilon_0}\vec{i}\left(\frac{-\frac{a}{d}q}{\left(d-\frac{a^2}{d}\right)^2} + \frac{q\left(1+\frac{a}{d}\right)}{d^2}\right) = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{-\frac{a}{d}\cdot d^2}{(d^2-a^2)^2} + \frac{(d+a)}{d^3}\right)\vec{i} =$$

$$\frac{q^2}{4\pi\varepsilon_0}\left(\frac{-aa^4+(d+a)(d^2-a^2)^2}{d^3(d^2-a^2)^2}\right)\vec{i} \Rightarrow \text{Erakarlea} \Longleftrightarrow \frac{-ad^4+(d+a)(d^2-a^2)^2}{d^3(d^2-a^2)^2} \leq 0$$

$d^3(d^2-a^2)^2 \geq 0$ beti $\rightarrow$ $-ad^4+(d+a)(d^2-a^2)^2 \leq 0$ $\rightarrow$ $(a^2+ad-d^2)(a^3-ad^2-d^3) \leq 0$

Esfera lurrera lotuta dagoenean. ($\phi(a,\theta)=0$)

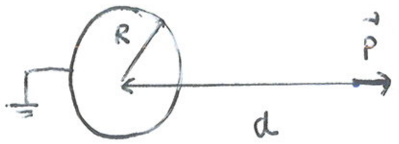



Problema baliokidean orain:

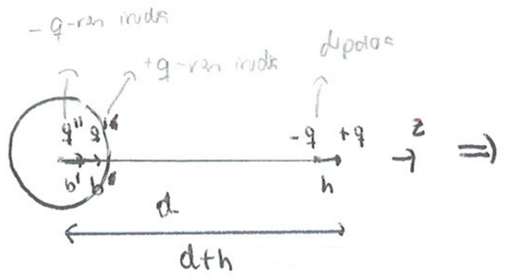
$$\vec{F} = \frac{q'q}{4\pi\varepsilon_0(d-b)^2}\vec{i} = \frac{-\frac{a}{d}q^2\vec{i}}{4\pi\varepsilon_0\left(d-\frac{a^2}{d}\right)^2} = \frac{-\frac{a}{\cancel{d}}q^2\vec{i}\cdot d^2}{4\pi\varepsilon_0(d^2-a^2)^2} =$$

$$\frac{-adq^2\vec{i}}{4\pi\varepsilon_0(d^2-a^2)^2}$$

1.22.)




Problema baliokidea $\longleftrightarrow$




$\Rightarrow$

Lurrera lotutako esfera eroalearen eta karga puntualaren probleman orain $\rightarrow$

$$q' = -(-q)\frac{R}{d} = q\frac{R}{d}, \quad b = \frac{R^2}{d} \quad \text{eta} \quad q'' = -q\frac{R}{(d+h)} \quad \text{eta} \quad b' = \frac{R^2}{(d+h)}$$


$\vec{p} = qh\hat{k}$

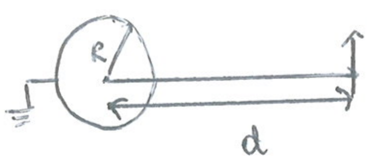
Indukitutako karga indizeko kargen baturaren berdina da → $Q = q' + q'' = q\frac{R}{d} - q\frac{R}{d+h} = qR\left(\frac{d+h-d}{d(d+h)}\right) = \frac{qRh}{d(d+h)} = \frac{pR}{d(d+h)} \xrightarrow{h\to 0} p\frac{R}{d^2}$

$\vec{p}$ dipoloan $+q$ eta $-q$-ren arteko distantzia

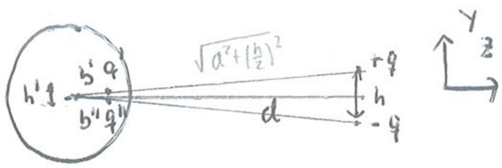
Gainera, $|\vec{p}'| = q'(b-b') = q\frac{R}{d}\left(\frac{R^2}{d} - \frac{R^2}{d+h}\right) = \frac{qR^3}{d}\left(\frac{d+h-d}{d(d+h)}\right) = qh\frac{R^3}{d^2(d+h)} = \frac{pR^3}{d^2(d+h)} \xrightarrow{h\to 0}$

$p\frac{R^3}{d^3} = p\left(\frac{R^3}{d^3}\right) \Rightarrow \vec{p}' = \vec{p}\left(\frac{R^3}{d^3}\right)$ (Norantza berean)

Bestalde, esferaren zentroa eta dipoloa lotzen duen ~~zentroratuta~~ zuzenarekiko perpendikular badago:



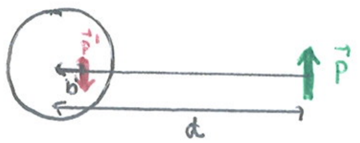

Problema baliokidea ⟷




Aurreko probleman oinarrituz (lurrari lotutako esfera eta karga puntuala): $\vec{p} = qh\hat{j}$

$\underset{h\to 0}{q' \simeq} -q\frac{R}{d}$ eta $\underset{h\to 0}{q'' \simeq} -(-q)\frac{R}{d} = q\frac{R}{d}$ eta $b' = \frac{R^2}{\sqrt{d^2+(\frac{h}{2})^2}} \xrightarrow{h\to 0} \frac{R^2}{d}$

$b'' = \frac{R^2}{\sqrt{d^2+(\frac{h}{2})^2}} \xrightarrow{h\to 0} \frac{R^2}{d} \Rightarrow$




$h' = \frac{b'h}{d} = \frac{R^2h}{d^2} = h\left(\frac{R}{d}\right)^2 \rightarrow \vec{p}' = -q''h'\hat{j} = -q\frac{R}{d}h\left(\frac{R}{d}\right)^2\hat{j} = -qh\hat{j}\left(\frac{R}{d}\right)^3 = -\vec{p}\left(\frac{R}{d}\right)^3$

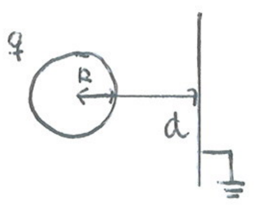
(aurkako noranzkoan)

$\frac{b'}{h'} = \frac{d}{h}$

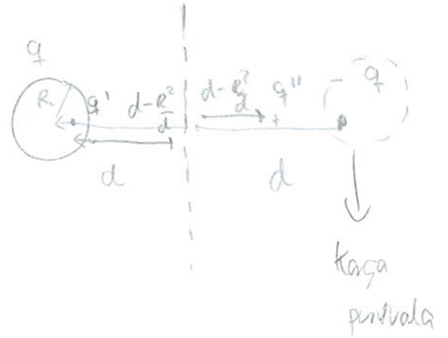
Eta esferan induzitutako karga indizeko kargen baturaren berdina denez,

$Q = q' + q'' = -q\frac{R}{d} + q\frac{R}{d} = 0$

1.23.)




q kargaz kargatutako R erradioko esfera eroalea eta lurrera konektatutako xafla eroale infinitua.




Esferaren indizioa planoaren eskuinaldean $-q$ karga da. Honela modu berean, indizioko karga bat izango du esferan (aurreko problema) eta horrek planoaren eskuinaldean beste bat. Honela infinituki...

$q \rightarrow -q$ planoan $\longrightarrow +q\frac{R}{d}$ zentrutik $\frac{R^2}{d}$ distantziara $\rightarrow -q\frac{R}{d}$ karga planotik $d-\frac{R^2}{d}$ distantziara (eskuinaldean) $\longrightarrow q\frac{R}{d}\left(\frac{R}{d-\frac{R^2}{d}}\right)$ karga zentrutik $\frac{R^2}{d-\frac{R^2}{d}}$ distantziara $\rightarrow$

Planoan induzitutako karga $\rightarrow$ $Q = -q - q\frac{R}{d} - qR\left(\frac{R}{d^2-R^2}\right) + \cdots$

Lorrain-Corson liburua (R-M-C)
↳ ekipotentzialak

1.24)


* λ luzerako karga dentsitateko lero batek sortzen duen potentziala.



$$E\cdot A=\frac{\lambda L}{\varepsilon_0}=E\cdot 2\pi r L \rightarrow \vec{E}=\frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}=-\vec{\nabla}\phi=-\frac{\partial\phi}{\partial r}\hat{r}\rightarrow$$

$$\phi(r)=\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_0}{r}\right) \qquad \phi(r_0)=0$$

* posizioa badakigu aurreko ariketaren ordez kalkulatu zehatz dagoen $\phi(C)$ $r_0$ kalkulatzeko

* Zilindroaren zentrutik b distantziara λ' luzerako karga dentsitateko lero irudikaria kokatuko dugu problema baliokidean. Potentziala $P(r,\varphi)$ $(r>R)$ puntuan honako izango da:

$$r_1=\sqrt{r^2+d^2-2rd\cos\varphi} \qquad r_2=\sqrt{r^2+b^2-2rb\cos\varphi}$$

$$\phi(r,\varphi)=\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_0}{r_1}\right)+\frac{\lambda'}{2\pi\varepsilon_0}\ln\left(\frac{r_0}{r_2}\right)=\frac{1}{4\pi\varepsilon_0}\left(\lambda\ln\left(\frac{r_0^2}{r_1^2}\right)+\lambda'\ln\left(\frac{r_0'^2}{r_2^2}\right)\right)=$$

$$\frac{1}{4\pi\varepsilon_0}\left(\lambda\ln\left(\frac{r_0^2}{r^2+d^2-2rd\cos\varphi}\right)+\lambda'\ln\left(\frac{r_0'^2}{r^2+b^2-2rb\cos\varphi}\right)\right)=\frac{1}{4\pi\varepsilon_0}\left(\ln\left(\frac{r_0^{2\lambda}}{(r^2+d^2-2rd\cos\varphi)^{\lambda}}\right)+\ln\left(\frac{r_0'^{2\lambda'}}{(r^2+b^2-2rb\cos\varphi)^{\lambda'}}\right)\right)=$$

$$\frac{1}{4\pi\varepsilon_0}\ln\left(\frac{r_0^{2\lambda}+r_0^{2\lambda'}}{(r^2+d^2-2rd\cos\varphi)^{\lambda}(r^2+b^2-2rb\cos\varphi)^{\lambda'}}\right) \; ; \; \phi(r\to\infty,\varphi)=0 \rightarrow \text{(Mugalde baldintza)}$$

$$\phi(r\to\infty,\varphi)=\frac{1}{4\pi\varepsilon_0}\ln\left(\frac{r_0^{2\lambda+2\lambda'}}{r^{2\lambda}\,r^{2\lambda'}}\right)=0 \longleftrightarrow (r_0)=r^{2\lambda+2\lambda'} \rightarrow 2(\lambda+\lambda')\ln r=0 \rightarrow$$

$2(\lambda+\lambda')\ln r_0$

$r\neq r_0$

~~$\ln r\neq 0$~~ $\quad \lambda+\lambda'=0 \rightarrow \lambda'=-\lambda$

simetria zilindrikoa betetzeko

$$\phi(r,\varphi)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{r_2^2}{r_1^2}\right)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{r^2+b^2-2rb\cos\varphi}{r^2+d^2-2rd\cos\varphi}\right),$$

$r=R$ denean $\phi(R,\varphi)=kte=V_0$ izan behar da, zilindroa eroale bat delako (ekipotentziala) $\rightarrow \phi(R,\varphi)=$

$$\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{R^2+b^2-2Rb\cos\varphi}{R^2+d^2-2Rd\cos\varphi}\right)=kte=V_0 \rightarrow \frac{4\pi\varepsilon_0 V_0}{\lambda}=\ln\left(\frac{R^2+b^2-2Rb\cos\varphi}{R^2+d^2-2Rd\cos\varphi}\right)$$

↗ ure bat, gainera $A^2$ jarri positiboa delako (esponentziala bati)

$$e^{\frac{4\pi\varepsilon_0 V_0}{\lambda}} = \frac{R^2+b^2-2Rb\cos\varphi}{R^2+d^2-2Rd\cos\varphi} = A^2 \rightarrow R^2+b^2-2Rb\cos\varphi = A^2R^2+A^2d^2-2RA^2d\cos\varphi \rightarrow$$

$$R^2(1-A^2)+b^2 = A^2d^2+2R(b-A^2d)\cos\varphi$$ → hau bete behar denez

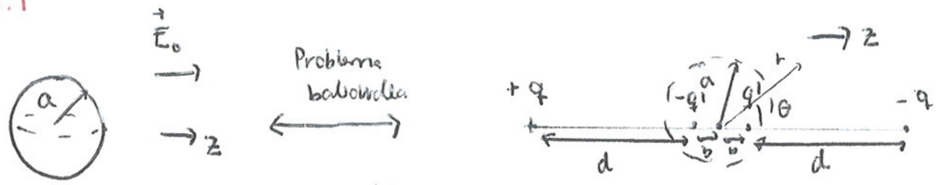
$\varphi$-ren edozein baliorako $b-A^2d=0$ izan behar da $\rightarrow A^2 = \frac{b}{d} \rightarrow$

$$R^2\left(1-\frac{b}{d}\right)+b^2 = \frac{b}{\cancel{d}}\cdot d^{\cancel{2}} \rightarrow b^2+R^2-b\frac{R^2}{d}-bd=0 \rightarrow b^2-b\left(\frac{R^2}{d}+d\right)+R^2=0$$

$$b = \frac{\left(\frac{R^2}{d}+d\right)\pm\sqrt{\left(\frac{R^2}{d}+d\right)^2-4R^2}}{2} = \frac{\left(\frac{R^2+d^2}{d}\right)\pm\sqrt{\frac{(R^2+d^2)^2-4R^2d^2}{d^2}}}{2} = \frac{R^2+d^2\pm\sqrt{R^4+d^4+2R^2d^2-4R^2d^2}}{2d} =$$

$$\frac{R^2+d^2\pm\sqrt{(R^2-d^2)^2}}{2d} = \frac{R^2+d^2\pm(R^2-d^2)}{2d} = \begin{cases} R^2/d \\ \cancel{d} \text{ erantzuna } <R \text{ izan behar da.}\end{cases}$$

1.26.)




↗ aurreko problema

Demagun eremu hori $+q$ eta $-q$ kargek sortutakoa dela $d\to\infty$ eramanez.

$q' = +\frac{a}{d}q$ $\quad b = \frac{a^2}{d}$ ; $d\to\infty$ $\quad b\to 0$ dipolo bat osatu.

$p = +q'\,2b = +q\frac{a}{d}\cdot\frac{2a^2}{d} = +2q\frac{a^3}{d^2}$ (falta zaigu $d$ deuseztea).

Potentziala edozein puntutan:

$$\phi(r,\theta) = \frac{-q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+d^2-2dr\cos\theta}} - \frac{1}{\sqrt{(dr/a)^2+a^2-2rd\cos\theta}} + \frac{1}{\sqrt{(dr/a)^2+a^2+2dr\cos\theta}} + \right.$$

$$\left. - \frac{1}{\sqrt{r^2+d^2+2dr\cos\theta}}\right] ,\ d\to\infty \rightarrow \phi(r,\theta) = \frac{-q}{4\pi\varepsilon_0}\left[\frac{1}{d\sqrt{1+\underbrace{\frac{r^2-2dr\cos\theta}{d^2}}_{\to 0}}} - \frac{1}{\frac{dr}{a}\sqrt{1+\underbrace{\frac{(a^2-2rd\cos\theta)a^2}{(dr)^2}}_{\to 0}}} + \frac{1}{\frac{dr}{a}\sqrt{1+\underbrace{\frac{(a^2+2dr\cos\theta)a^2}{d^2r^2}}_{\to 0}}}\right.$$

$$\left. - \frac{1}{d\sqrt{1+\underbrace{\frac{r^2+2dr\cos\theta}{d^2}}_{\to 0}}}\right] \underset{\downarrow}{=} \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{d}\left(1-\frac{\cancel{r^2}-2dr\cos\theta}{2d^2}\right) - \frac{a}{dr}\left(1-\frac{\cancel{a^4}-2rda^2\cos\theta}{2d^2r^2}\right) + \frac{a}{dr}\left(1-\frac{\cancel{a^4}+2dra^2\cos\theta}{2d^2r^2}\right) + \right.$$

$$\left[\frac{1}{\sqrt{1+x}} \simeq 1-x/2 \quad (x<<0)\right] \qquad \left. - \frac{1}{d}\left(1-\frac{\cancel{r^2}+2dr\cos\theta}{2d^2}\right)\right] =$$

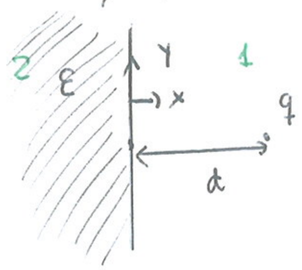
$$\frac{-q}{4\pi\varepsilon_0}\left(\frac{\cancel{2}r\cos\theta}{\cancel{2}d^2}-\frac{\cancel{2}r d a^3\cos\theta}{\cancel{2}d^2r^2dr}-\frac{\cancel{2}a^3\cos\theta}{\cancel{2}d^2r^2}+\frac{r\cos\theta}{d^2}\right)=\frac{-q\cdot 2}{4\pi\varepsilon_0 d^2}\left(r\cos\theta-\frac{a^3}{r^2}\cos\theta\right)=$$

$$\frac{-q}{2\pi\varepsilon_0 d^2}\left(r-\frac{a^3}{r^2}\right)\cos\theta \quad , \quad \text{Beraz} \quad r\to\infty \quad \phi(r,\theta)=-E_0 r\cos\theta=\frac{-q}{2\pi\varepsilon_0 d^2}r\cos\theta \rightarrow$$

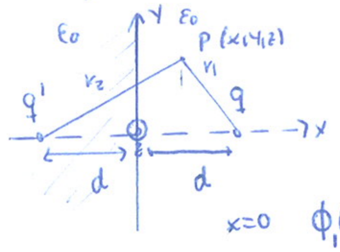
$$E_0=+\frac{q}{2\pi\varepsilon_0 d^2}\rightarrow +\frac{q}{d^2}=+2\pi\varepsilon_0 E_0 \longleftrightarrow p=+4\pi\varepsilon_0 E_0 a^3 \; , \; \vec{p}=4\pi\varepsilon_0 a^3\vec{E}_0 \Rightarrow$$

$$\text{Eremua} \rightarrow \phi(\vec{r})=-E_0 r\cos\theta+\frac{+a^3 q}{2\pi\varepsilon_0 d^2 r}=-E_0 r\cos\theta+\underbrace{\frac{a^3}{r^2}E_0\cos\theta}_{=\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$$

1.27.) → plano dielektriko infinitoa




1) • $x \geq 0$ problema baliokidea:

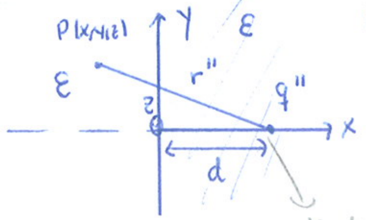



$r_1=\sqrt{(x-d)^2+\underbrace{y^2+z^2}_{R^2}}$ , $r_2=\sqrt{(x+d)^2+\underbrace{y^2+z^2}_{R^2}}$

$x=0 \rightarrow r_1=r_2=\sqrt{d^2+R^2}$

$x=0 \quad \phi_1(R)=\dfrac{q+q'}{4\pi\varepsilon_0\sqrt{d^2+R^2}}$ ; $\phi_1(x,R)=\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q}{r_1}+\dfrac{q'}{r_2}\right)$

2). $x \leq 0$ problema baliokidea.




$r''=\sqrt{(x-d)^2+y^2+z^2}=\sqrt{(x-d)^2+y^2+z^2}$ ; $x=0 \rightarrow r''=r_1=r_2=\sqrt{d^2+R^2}$

$\phi_2(R)=\dfrac{q''}{4\pi\varepsilon\sqrt{d^2+R^2}} \quad x=0$ ; $\phi_2(x,R)=\dfrac{1}{4\pi\varepsilon}\dfrac{q''}{r''}$

Kontsideratzen ari garen beste aldean jarri behar dielektrikoa (?) legoke, bestela ρ aldatuko litzateke eta poissonen ekuazioa ez litzateke beteko inorgo

Mugalde baldintzak:

$\phi_1(R,x)=\phi_2(R,x)\big|_{x=0} \rightarrow \dfrac{q''}{\cancel{4\pi}\varepsilon\cancel{\sqrt{d^2+R^2}}}=\dfrac{q+q'}{\cancel{4\pi}\varepsilon_0\cancel{\sqrt{d^2+R^2}}} \rightarrow \varepsilon_0 q''=\varepsilon(q+q') \rightarrow q''=\dfrac{\varepsilon}{\varepsilon_0}(q+q')$

$\varepsilon_0 E_{1x}=\varepsilon E_{2x}\big|_{x=0} \rightarrow E_{1x}=-\dfrac{\partial\phi_1}{\partial x}=\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q(x-d)}{((x-d)^2+R^2)^{3/2}}+\dfrac{q'(x+d)}{((x+d)^2+R^2)^{3/2}}\right)$

$E_{2x}=-\dfrac{\partial\phi_2}{\partial x}=\dfrac{q''(x-d)}{((x-d)^2+R^2)^{3/2}}\cdot\dfrac{1}{4\pi\varepsilon} \rightarrow \varepsilon_0 E_{1x}\big|_{x=0}=\dfrac{1}{\cancel{4\pi}}\left(\dfrac{-dq}{(d^2+\cancel{R^2})^{3/2}}+\dfrac{q'd}{(d^2+\cancel{R^2})^{3/2}}\right)=$

$\dfrac{1}{\cancel{4\pi}}\dfrac{-q''d}{(d^2+\cancel{R^2})^{3/2}} \Rightarrow -q''=-q+q' \rightarrow q''=q-q'=\dfrac{\varepsilon}{\varepsilon_0}(q+q')\rightarrow$

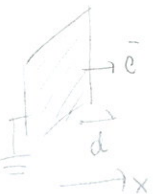
$q\left(1-\dfrac{\varepsilon}{\varepsilon_0}\right)=q'\left(\dfrac{\varepsilon}{\varepsilon_0}+1\right)\rightarrow q'=q\dfrac{(\varepsilon_0-\varepsilon)}{(\varepsilon+\varepsilon_0)}$ , $q''=q\dfrac{2\varepsilon}{\varepsilon+\varepsilon_0}$

$\varepsilon \to \infty \quad q' = -q$ (plano eroalearen kasua)

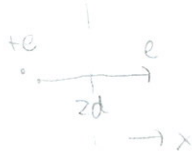
Falta da 1.25.

1.16.)

Deskargatuta:




$\to$ Problema baliokidea $\to$




$$\vec{F} = \frac{-e^2}{4\pi\varepsilon_0 (2d)^2}\hat{i} = \frac{-e^2}{16\pi\varepsilon_0 d^2}\hat{i}$$

Indar aldakorra $\to$ ilusiko duen potentziala: $\phi(x,y,z) = \dfrac{e}{4\pi\varepsilon_0\sqrt{(x+d)^2+y^2+z^2}}$

d izanik $\bar{e}$-eta eroalearen arteko distantzia.

EZ (ez dago ilusiko)

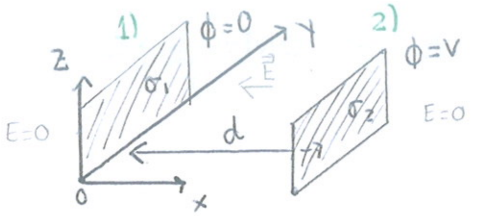
# ELEKTROMAGNETISMOA II:

16-09-17

## 1. EREMU ESTATIKOETARAKO MUGA-PROBLEMAK:

### POISSON ETA LAPLACE EKUAZIOAK ELEKTROSTATIKAN

1.1)




Simetria → z eta y-ren independentea → x-ren menpekoa soilik.

$$\text{M. B}\begin{cases}\phi(0)=0\\ \phi(d)=V\\ \phi \text{ finitua } \forall x\end{cases}$$

- $x \leq 0$ $\quad \nabla^2\phi=0 \rightarrow \dfrac{d^2\phi}{dx^2}=0 \rightarrow \dfrac{d\phi}{dx}=A \rightarrow \phi=Ax+B$

Mugalde baldintzak: $\phi(0)=B=0$

$\phi$ finitua $\forall x \leftrightarrow A=0$

$$\left.\begin{matrix}\phi(0)=B=0\\ \phi \text{ finitua } \forall x \leftrightarrow A=0\end{matrix}\right\} \phi(x)=0$$

- $x \geq d$ $\quad \nabla^2\phi=0 \rightarrow \dfrac{d^2\phi}{dx^2}=0 \rightarrow \dfrac{d\phi}{dx}=A \rightarrow \phi=Ax+B$

Mugalde baldintzak: $\phi(d)=Ad+B=V \rightarrow B=V-Ad=V$

$\phi$ finitua $\forall x \leftrightarrow A=0$

$$\left.\begin{matrix}\phi(d)=Ad+B=V \rightarrow B=V-Ad=V\\ \phi \text{ finitua } \forall x \leftrightarrow A=0\end{matrix}\right\} \phi(x)=V$$

- $0 \leq x \leq d$ $\quad \nabla^2\phi=0 \rightarrow \dfrac{d^2\phi}{dx^2}=0 \rightarrow \dfrac{d\phi}{dx}=A \rightarrow \phi=Ax+B$

Mugalde baldintzak: $\phi(0)=B=0$

$\phi(d)=A\cdot d=V \rightarrow A=\dfrac{V}{d}$

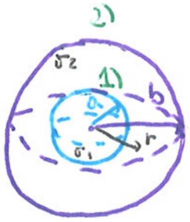
$$\left.\begin{matrix}\phi(0)=B=0\\ \phi(d)=A\cdot d=V \rightarrow A=\frac{V}{d}\end{matrix}\right\} \phi(x)=\frac{V}{d}x$$

$$\phi(x)=\begin{cases}0 & x<0\\ \dfrac{V}{d}x & 0\leq x<d\\ V & x\geq d\end{cases} \rightarrow \vec{E}=-\vec{\nabla}\phi=\begin{cases}0 & x<0\\ -\dfrac{V}{d}\vec{x} & 0\leq x<d\\ 0 & x\geq d\end{cases}$$

1 → $\sigma_1 = \varepsilon_0 \left( \vec{E}(0)\cdot \hat{n}_1 \right) = -\varepsilon_0 |\vec{E}(0)| = -\varepsilon_0 \dfrac{V}{d}$ $\qquad \hat{n}_1 = \vec{i}$

2 → $\sigma_2 = \varepsilon_0 \left( \vec{E}(d)\cdot \hat{n}_2 \right) = \varepsilon_0 |\vec{E}(d)| = \varepsilon_0 \dfrac{V}{d}$ $\qquad \hat{n}_2 = -\vec{i}$

1.2)




$\varphi$ eta $\theta$-ran independentea → r-ran menpekoa soilik (simetriagatik)

$$\text{M.B} \begin{cases} \phi(a)=0 \\ \phi(b)=V \\ \phi(\infty)=0 \end{cases}$$

- $r \le a$ $\qquad \nabla^2\phi = 0 \rightarrow \dfrac{d}{dr}\left( r^2 \dfrac{d\phi}{dr} \right) = 0 \rightarrow r^2\dfrac{d\phi}{dr} = A \rightarrow \dfrac{d\phi}{dr} = \dfrac{A}{r^2} \rightarrow \phi(r) = \dfrac{B}{r} + C$

Mugalde baldintzak: $\phi(a) = \dfrac{B}{a} + C = 0 \rightarrow C = -\dfrac{B}{a}$

Eroalearen barnean ez dago eremurik → gai monopolarra 0 izan behar da → $B = 0$ $\Big\}\ \phi(r) = 0$

→ ez dago kargarik

- $a \le r \le b$ $\qquad \nabla^2\phi = 0 \rightarrow \phi(r) = \dfrac{B}{r} + C$

Mugalde baldintzak: $\phi(a) = \dfrac{B}{a} + C = 0 \rightarrow C = -\dfrac{B}{a}$

$\phi(b) = \dfrac{B}{b} + C = V \rightarrow C = V - \dfrac{B}{b} = -\dfrac{B}{a} \rightarrow V = B\left(\dfrac{1}{b} - \dfrac{1}{a}\right) = B\left(\dfrac{a-b}{ab}\right) \rightarrow V\cdot\dfrac{ab}{a-b} = B$ $\Big\} \Rightarrow$

$$\phi(r) = V\frac{ab}{a-b}\left(\frac{1}{r} - \frac{1}{a}\right)$$

- $r \geq b$ $\qquad \nabla^2\phi = 0 \rightarrow \dfrac{d}{dr}\left( r^2 \dfrac{d\phi}{dr} \right) = 0 \rightarrow \phi(r) = \dfrac{B}{r} + C$

Mugalde baldintzak: $\phi(b) = \dfrac{B}{b} + C = V \rightarrow C = V - \dfrac{B}{b} \rightarrow B = V\cdot b$

$\phi(\infty) = C = 0$ $\qquad \Big\}\ \phi(r) = V\cdot\dfrac{b}{r}$

$$\phi(r) = \begin{cases} 0 & a > r \\ V\dfrac{ab}{a-b}\left(\dfrac{1}{r} - \dfrac{1}{a}\right) & a \le r < b \\ V\dfrac{b}{r} & r \geq b \end{cases} \rightarrow \vec{E} = -\vec{\nabla}\phi = \begin{cases} 0 & a > r \\ V\dfrac{ab}{a-b}\cdot\dfrac{1}{r^2}\,\hat{r} & a \le r < b \\ V\dfrac{b}{r^2}\,\hat{r} & r \geq b \end{cases}$$

1 → $\sigma_1 = \vec{D}_2 - \vec{D}_1\Big|_{r=a}\cdot \hat{n}_1 = \varepsilon_0 (E_2 - E_1)\Big|_{r=a}\cdot\hat{n}_1 = \varepsilon_0 \cdot V\dfrac{\cancel{a}b}{a-b}\cdot\dfrac{\hat{r}\cdot\hat{r}}{a^{\cancel{2}}} = \dfrac{\varepsilon_0}{a}\,\dfrac{V\cdot b}{a-b}$


$Q_1 = 4\pi a^2 \sigma_1 = \varepsilon_0 \dfrac{V a b}{a-b}\cdot 4\pi$

↳ normalak

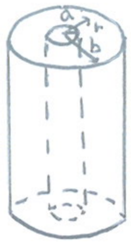
2→ $\sigma_2 = \vec{D}_2 - \vec{D}_1 \Big|_{r=b} \hat{n}_2 = \varepsilon_0 (E_2 - E_1)\Big|_{r=b} \hat{n}_2 = \varepsilon_0 \left( V \cdot \frac{b}{b^2} - \frac{V \cdot ab}{a-b} \cdot \frac{1}{b^2} \right) \hat{r}\cdot\hat{r} = \varepsilon_0 \frac{V}{b}\left(1 - \frac{a}{a-b}\right) = \frac{\varepsilon_0 \cdot V}{b-a}$

$$Q_2 = 4\pi b^2 \cdot \sigma_2 = 4\pi\varepsilon_0 \frac{V\cdot b^2}{b-a}$$

(Gauss eginez gauza bera)

13.)

$z$ eta $\varphi$-ren independentzea simetriagatik → $r$-ren menpekoa soilik



$$\text{M.B} \begin{cases} \phi(a) = 0 \\ \phi(b) = V \\ \phi \text{ finitua } \forall r \end{cases}$$

- $r \leq a$ $\quad \nabla^2\phi = 0 \rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0 \rightarrow r\frac{d\phi}{dr} = A \rightarrow \phi(r) = B\ln r + C$

Mugalde baldintzak: $\phi(a) = B\ln a + C = 0 \rightarrow C = -B\ln a = 0$

$\phi$ Finitua $\forall r \rightarrow \ln r$ $\;r=0$-n ez dago definituta → $B = 0$

↳ bestela ez dago kargarik

$\Big\}\ \phi(r) = 0$

- $a \leq r \leq b$ $\quad \nabla^2\phi = 0 \rightarrow \phi(r) = B\ln r + C$

Mugalde baldintzak: $\phi(a) = B\ln a + C = 0 \rightarrow C = -B\ln a$

$\phi(b) = B\ln b + C = B\ln b - B\ln a = B(\ln b - \ln a) = B\ln\left(\frac{b}{a}\right) = V \rightarrow B = \frac{V}{\ln\left(\frac{b}{a}\right)}$ $\Big\} \Rightarrow$

$$\phi(r) = \frac{V}{\ln\left(\frac{b}{a}\right)}(\ln r - \ln a) = V \cdot \frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)}$$

- $r \geq b$ $\quad \nabla^2\phi = 0 \rightarrow \phi(r) = B\ln r + C$

Mugalde baldintzak: $\phi(b) = B\ln b + C = V \rightarrow C = V - B\ln b = V$

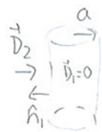
$\phi$ Finitua $\forall r \rightarrow \ln r \underset{r\to\infty}{\rightarrow} \infty \leftrightarrow B = 0$

$\Big\}\ \phi(r) = V$

$$\phi(r) = \begin{cases} 0 & r < a \\ \frac{V}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{r}{a}\right) & a \leq r < b \\ V & r \geq b \end{cases} \rightarrow \vec{E} = -\vec{\nabla}\phi = \begin{cases} 0 & a > r \\ -\frac{V}{\ln\left(\frac{b}{a}\right)}\cdot\frac{1}{r}\,\hat{r} & a \leq r < b \\ 0 & r \geq b \end{cases}$$

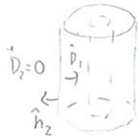
$1 \to \sigma_1 = \vec{D}_2 - \vec{D}_1 \big|_{r=a} \cdot \hat{n}_1 = \varepsilon_0 (\vec{E}_2 - \vec{E}_1)\big|_{r=a} \cdot \hat{n}_1 = \varepsilon_0 \left( \frac{-V}{\ln|\frac{b}{a}|} \cdot \frac{1}{a} \hat{\rho} \right) \cdot \hat{\rho} = - \varepsilon_0 \frac{V}{\ln|\frac{b}{a}|} \cdot \frac{1}{a}$

$\frac{Q_1}{L} = \sigma_1 \cdot 2\pi a = -2\pi \varepsilon_0 \frac{V}{\ln|\frac{b}{a}|}$

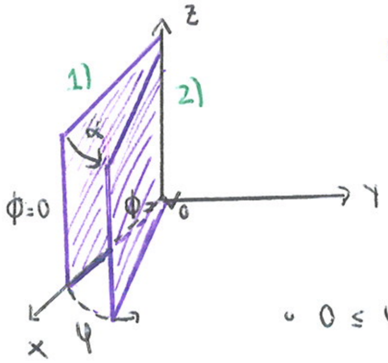



$2 \to \sigma_2 = \vec{D}_2 - \vec{D}_1 \big|_{r=b} \cdot \hat{n}_2 = \varepsilon_0 (\vec{D}_2 - \vec{D}_1)\big|_{r=b} \cdot \hat{n}_2 = \varepsilon_0 \cdot \left( \frac{V}{\ln|\frac{b}{a}|} \cdot \frac{1}{b} \cdot \hat{\rho} \right) \cdot \hat{\rho} = \varepsilon_0 \cdot \frac{V}{\ln|\frac{b}{a}|} \cdot \frac{1}{b}$

$\frac{Q_2}{L} = \sigma_2 \cdot 2\pi b = 2\pi V \cdot \frac{\varepsilon_0}{\ln|\frac{b}{a}|}$




## 1.4)




Simetriagatik z eta ρ-ran independentea (infinitutzat hartu) → φ-ran menpekoa soilik

$$\text{M.B} \begin{cases} \phi(0) = 0 \\ \phi(\alpha) = V_0 \\ \text{Periodikoa}, \ \phi(\varphi + 2\pi) = \phi(\varphi) \qquad \varphi \geq \alpha \end{cases}$$

- $0 \leq \varphi < \alpha$ $\qquad \nabla^2 \phi = \frac{1}{\rho^2} \frac{d^2\phi}{d\varphi^2} = 0 \to \frac{d^2\phi}{d\varphi^2} = 0 \to \frac{d\phi}{d\varphi} = A \to \phi = A\varphi + B$

Mugalde baldintzak: $\phi(0) = B = 0$

$\phi(\alpha) = A \cdot \alpha = V_0 \to A = \frac{V_0}{\alpha}$ $\Big\} \ \phi(\varphi) = V_0 \cdot \frac{\varphi}{\alpha}$

- $\alpha \leq \varphi < 2\pi$ $\qquad \nabla^2 \phi = 0 \to \phi = A\varphi + B$

Mugalde baldintzak: $\phi(\varphi + 2\pi) = A(\varphi + 2\pi) + B = \cancel{A\varphi} + 2A\pi + \cancel{B} = \phi(\varphi) = \cancel{A\varphi} + \cancel{B} \to$

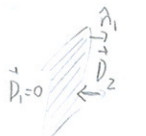
$A = 0$

$\phi(\alpha) = B = V_0$ $\Big\} \ \phi(\varphi) = V_0$

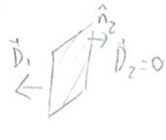
$\vec{\nabla}\phi = \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\hat{\varphi}$

$$\phi(\varphi) = \begin{cases} V_0 \cdot \frac{\varphi}{\alpha} & 0 \leq \varphi < \alpha \\ V_0 & \alpha \leq \varphi < 2\pi \end{cases} \to \vec{E} = -\vec{\nabla}\phi = \begin{cases} -\frac{V_0}{\rho\alpha} \hat{\varphi} & 0 \leq \varphi < \alpha \\ 0 & \alpha \leq \varphi < 2\pi \end{cases}$$

$1 \to \sigma_1 = \vec{D}_2 - \vec{D}_1 \big|_{\varphi=0} \cdot \hat{n}_1 = \varepsilon_0 (\vec{E}_2 - \vec{E}_1)\big|_{\varphi=0} \cdot \hat{n}_1 = \varepsilon_0 \left( -\frac{V_0}{\alpha\rho} \hat{\varphi} \right) \cdot \hat{\varphi} = -\varepsilon_0 \frac{V_0}{\rho\alpha}$

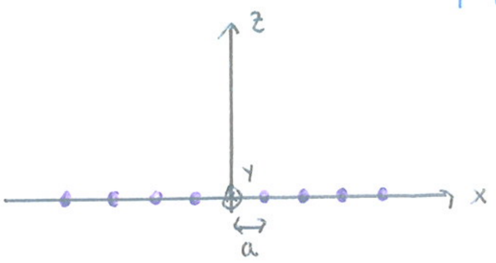



$2 \to \sigma_2 = \vec{D}_2 - \vec{D}_1 \big|_{\varphi=\alpha} \cdot \hat{n}_2 = \varepsilon_0 (\vec{E}_2 - \vec{E}_1)\big|_{\varphi=\alpha} \cdot \hat{n}_2 = \varepsilon_0 \left( \frac{V_0}{\rho\alpha} \hat{\varphi} \right) \cdot \hat{\varphi} = \varepsilon_0 \frac{V_0}{\rho\alpha}$




* edo $\sigma_1 = \varepsilon_0 \vec{E}\big|_{\varphi=0} \cdot \hat{n}_1$, $\sigma_2 = \varepsilon_0 \vec{E}\big|_{\varphi=\alpha} \cdot \hat{n}_2$

1.5)




Y norabidean hari amaigabeak → Y-ren independentea → x eta z-ren menpekoa soilik

$$\begin{cases} \phi(x,\infty)=0 \\ \text{Periodikoa } \phi(x+a,z)=\phi(x,z) \\ \dfrac{\partial\phi}{\partial x}(0,z)=0 \quad \text{(Grafikotik)} \end{cases}$$

$\nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial z^2}=0$ Ald. banantzea $\phi(x,z)=X(x)\cdot Z(z) \rightarrow \nabla^2\phi = Z\cdot X'' + X\cdot\ddot{Z}=0 \rightarrow$

$\dfrac{X''}{X}+\dfrac{\ddot{Z}}{Z}=0 \rightarrow -\dfrac{X''}{X}=\dfrac{\ddot{Z}}{Z}=K$

$\phi(x+a,z)=X(x+a)Z(z)=\phi(x,z)=X(x)Z(z) \leftrightarrow$
$\uparrow$ $X(x+a)=X(x)$

1• Demagun $K=0 \rightarrow$ • $X''=0 \rightarrow X=Ax+B \rightarrow$ periodikoa $\leftrightarrow A=0$

• $\ddot{Z}=0 \rightarrow Z=Az+B$, $\phi(x,\infty)=X(x)Z(\infty)=0 \rightarrow Z(\infty)=0$ ez da betetzen $\rightarrow$

$K\neq 0$

2• Demagun $K<0 \rightarrow$ • $X''+KX=0 \rightarrow X=Ae^{\sqrt{-K}x}+Be^{-\sqrt{-K}x}$, $X(x)=X(x+a)$ $0=0$

• $\ddot{Z}-KZ=0 \rightarrow Z=A\cos(\sqrt{-K}z)+B\sin(\sqrt{-K}z)$ $Z(\infty)=0 \leftrightarrow A=B=0$

Soluzio tribiala $K>0$ beraz

3• $K>0 \rightarrow X''+KX=0 \rightarrow$ • $X=A\sin(\sqrt{K}x+\delta)$, $X(x)=X(x+a) \rightarrow$ Periodikoa $\rightarrow$

$X(x+a)=A\sin(\sqrt{K}x+\sqrt{K}a+\delta)=X(x)=A\sin(\sqrt{K}x+\delta) \rightarrow$

$[\sin(\sqrt{K}x+\sqrt{K}a+\delta)=\sin(\sqrt{K}x+\delta)\cos(\sqrt{K}a)+\cos(\sqrt{K}x+\delta)\sin(\sqrt{K}a)=\sin(\sqrt{K}x+\delta)]$

$\cancel{\sqrt{K}x}+\sqrt{K}a+\cancel{\delta}=\cancel{\sqrt{K}x}+\cancel{\delta}+2\pi n \rightarrow \sqrt{K}=\dfrac{2\pi n}{a}$ $n\in\mathbb{N}$ *

• $Z''-KZ=0 \rightarrow Z=Ae^{\sqrt{K}z}+Be^{-\sqrt{K}z} \rightarrow Z(\infty)=0 \leftrightarrow A=0$

* $\dfrac{\partial\phi}{\partial x}(0,z)=X'(0)Z(z)=0 \leftrightarrow X'(0)=0 \rightarrow X=A\sin(\sqrt{K}x+\delta)$, $X'=A\sqrt{K}\cos(\sqrt{K}x+\delta)$

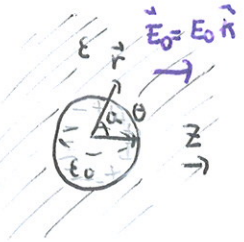
$X'(0)=A\sqrt{K}\cos(\delta)=0 \rightarrow \delta=\dfrac{2m+1}{2}\pi \rightarrow$ hartu $\underset{\downarrow\, m=0}{\delta=\dfrac{\pi}{2}} \rightarrow X=A\cos(\sqrt{K}x)$

A eta B n-ren menpekoak

$$\text{Beraz} \rightarrow \phi(x,z) = X(x)Z(z) = A\cos(\sqrt{k}x)\cdot Be^{-\sqrt{k}z} = A_n e^{-\frac{2\pi n}{a}z}\cdot\cos\left(\frac{2\pi n x}{a}\right)$$

$$\sqrt{k} = \frac{2\pi n}{a}$$

1.8.)




$\phi$-ren independentzia simetriagatik $\rightarrow$ $\theta$ eta $r$-ren menpekoa soilik

$$\vec{E} = \vec{E}_0 + \vec{E}_{esf.} \; ; \; \vec{E}(r,\theta)\Big]_{r\to\infty} = \vec{E}_0 = E_0\hat{k} = -\vec{\nabla}\phi \leftrightarrow \phi(r,\theta)\Big]_{r\to\infty} = -E_0 r\cos\theta$$

$$* \; r > a \rightarrow \nabla^2\phi = 0 \rightarrow \phi(r,\theta) = \sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+1)}\right]P_n(\cos\theta) =$$

$$A_0 + \frac{B_0}{r} + A_1 r\cos\theta + \frac{B_1}{r^2}\cos\theta + \frac{1}{2}A_2 r^2(3\cos^2\theta - 1) + \frac{1}{2}\frac{B_2}{r^3}(3\cos^2\theta - 1) + \ldots$$

$$\phi(r,\theta)\Big]_{r\to\infty} = -E_0 r\cos\theta \rightarrow A_1 = -E_0 \qquad A_n = 0 \quad \forall n \in \mathbb{N} - \{1\}$$

$$\phi(\bar{a},\theta) = \phi(a^+,\theta) \; *$$

$B_n = 0 \quad \forall n \in \mathbb{N} - \{1\} \rightarrow B_0 = 0$ gai monopolarra 0 izan behar delako, karga netoa 0 delako.

$$\phi(r,\theta) = -E_0 r\cos\theta + \frac{B_1}{r^2}\cos\theta$$

$$* \; r < a \rightarrow \nabla^2\phi = 0 \rightarrow \phi(r,\theta) = \tilde{A}_1 r\cos\theta + \frac{\tilde{B}_1}{r^2}\cos\theta$$

$\phi$ finitua da jatorrian (ez dago kargarik) $\rightarrow \tilde{B}_1 = 0$ (Bestela $r\to 0$ infinitura doa)

$$\phi(r,\theta) = \tilde{A}_1 r\cos\theta$$

$$* \; \phi(r,\theta) = \begin{cases} -E_0 r\cos\theta + \frac{B_1}{r^2}\cos\theta & r \geq a \\ \tilde{A}_1 r\cos\theta & r \leq a \end{cases} \Rightarrow \text{jarraitua izan behar da } r = a \text{ puntuan}$$

$n = 1 \rightarrow E_0 a = B_1 \cdot a^{-2} - \tilde{A}_1 a$

$$* \sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta) - E_0 a\cos\theta = \sum_{n=0}^{\infty}\tilde{A}_n a^n P_n(\cos\theta) \rightarrow E_0 a\cos\theta = \sum_{n=0}^{\infty}\left(B_n a^{-(n+1)} - \tilde{A}_n a^n\right)P_n(\cos\theta)$$

$$-E_0 a\cos\theta + \frac{B_1}{a^2}\cos\theta = \tilde{A}_1 a\cos\theta \rightarrow -E_0 a + \frac{B_1}{a^2} = \tilde{A}_1 a \rightarrow B_1 = a^3(\tilde{A}_1 + E_0)$$

* Gainazalean ez dago kargarik (askeak) → $D_{1r} = D_{2r}\Big]_{r=a}$ (erradialak) → $\varepsilon_0\left(\frac{-\partial\phi}{\partial r}\right)_{r=a^-} = \varepsilon\left(\frac{-\partial\phi}{\partial r}\right)_{r=a^+}$

anulatu $B_n / A_n$

$$\varepsilon\cdot\left(E_0\cos\theta + \frac{2B_1}{r^3}\cos\theta\right)\Big|_{r=a} = -\varepsilon_0\tilde{A}_1\cos\theta\Big]_{r=a} \rightarrow \varepsilon\left(E_0\cos\theta + \frac{2B_1}{a^3}\cos\theta\right) = -\varepsilon_0\tilde{A}_1\cos\theta$$

$$\varepsilon\left(E_0+2\frac{B_1}{a^3}\right)=-\varepsilon_0\tilde{A}_1 \rightarrow \varepsilon\left(E_0+2\frac{a^3}{a^3}(\tilde{A}_1+E_0)\right)=-\varepsilon_0\tilde{A}_1 = \varepsilon\left(3E_0+2\tilde{A}_1\right)=-\varepsilon_0\tilde{A}_1 \rightarrow$$

$$3\varepsilon E_0+2\varepsilon\tilde{A}_1=-\varepsilon_0\tilde{A}_1 \rightarrow \tilde{A}_1(2\varepsilon+\varepsilon_0)=-3\varepsilon E_0 \rightarrow \tilde{A}_1=\frac{-3\varepsilon E_0}{2\varepsilon+\varepsilon_0}=\frac{-3E_0}{2+\frac{\varepsilon_0}{\varepsilon}}=\frac{-3E_0}{2+\frac{1}{\varepsilon_r}}=$$

$$\frac{-3E_0\,\varepsilon_r}{2\varepsilon_r+1}, \quad B_1=a^3\left(-\frac{3E_0\varepsilon_r}{2\varepsilon_r+1}+E_0\right)=a^3\left(\frac{-3E_0\varepsilon_r+2E_0\varepsilon_r+E_0}{2\varepsilon_r+1}\right)=$$

$$a^3\frac{(E_0-E_0\varepsilon_r)}{2\varepsilon_r+1}=E_0a^3\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}$$

$$\text{Beraz} \rightarrow \phi(r,\theta)=\begin{cases}\underbrace{-E_0r\cos\theta}_{\vec{E}_0}+\underbrace{\frac{E_0a^3}{r^2}\cos\theta\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}}_{\vec{E}_{esfera}\ *} & r\geq a\\[2ex] -\frac{3E_0\varepsilon_r}{2\varepsilon_r+1}r\cos\theta & r\leq a\end{cases}$$

$$*\ \vec{E}_{esf}=\frac{E_0a^3}{r^2}\cos\theta\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}=\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}a^3\frac{\vec{E}_0\cdot\vec{r}}{r^3}=\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0r^3} \longleftrightarrow \vec{p}=4\pi\varepsilon_0a^3\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}\vec{E}_0$$

$$\vec{P}=\frac{\vec{p}}{V}=\frac{\cancel{4\pi}\varepsilon_0\cancel{a^3}}{\frac{\cancel{4\pi}\cancel{a^3}}{3}}\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}\vec{E}_0=3\varepsilon_0\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}\vec{E}_0$$

$$\vec{E}=-\vec{\nabla}\phi=\begin{cases}\frac{3E_0\varepsilon_r}{2\varepsilon_r+1}(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta})=\frac{3E_0\varepsilon_r}{2\varepsilon_r+1}\hat{k} & r<a\\[2ex] E_0\hat{k}+\frac{E_0a^3}{r^3}\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}) & r\geq a\\ * & \end{cases}$$

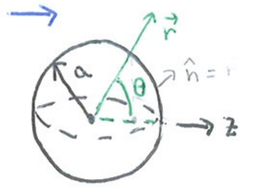
$$\parallel$$

$$-\left(\frac{\partial\phi}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}\right)$$

$$*\ \underbrace{E_0\cos\theta\,\hat{r}-E_0\sin\theta\,\hat{\theta}}_{E_0\hat{k}\,=\,\vec{E}_0}+\frac{2E_0a^3}{r^3}\cos\theta\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}\hat{r}+\frac{E_0a^3}{r^3}\sin\theta\frac{(1-\varepsilon_r)}{2\varepsilon_r+1}\hat{\theta}$$

1.9.1

$\vec{P}_0 = P_0\,\hat{n}$ $\quad \sigma_p = \vec{P}_0 \cdot \hat{n} = P_0 \cos\theta \qquad \phi$ finitua $\forall r \in [0, +\infty)$, $\varphi$-ren independentea




• $r \geq a$ $\qquad \nabla^2\phi = 0 \nearrow \quad \phi_1(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\theta)$

$A_n = 0 \rightarrow \phi$ finitua (bestela $r \to \infty$ $\phi \to \infty$) $\Rightarrow$

$$\phi_1(r,\theta) = \sum_{n=0}^{\infty} B_n \frac{1}{r^{n+1}} P_n(\cos\theta)$$

• $r < a$ $\qquad \nabla^2\phi = 0 \rightarrow \quad \phi_2(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\theta)$

$B_n = 0 \rightarrow \phi$ finitua (bestela jatorrian infinitua) $\Rightarrow$

$$\phi_2(r,\theta) = \sum_{n=0}^{\infty} A_n \cdot r^n P_n(\cos\theta)$$

Mugalde baldintzak $\Rightarrow$ * $\phi(r,\theta)$ jarraitua $\quad r=a \rightarrow \phi_1(a,\theta) = \phi_2(a,\theta) \rightarrow$

$$\sum_{n=0}^{\infty} B_n \cdot \frac{1}{a^{n+1}} P_n(\cos\theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) \rightarrow \frac{B_n}{a^{n+1}} = A_n a^n \rightarrow B_n = A_n a^{2n+1} \quad *'$$

$\sigma = \sigma_p + \sigma_j = \sigma_p$ ($\sigma_j \to$ askea)

$$* \quad \vec{E}_1 - \vec{E}_2 \Big|_{r=a} \hat{n} = \frac{\sigma_p}{\varepsilon_0} = -\left[\frac{\partial\phi_1}{\partial r} - \frac{\partial\phi_2}{\partial r}\right]\Big|_{r=a} = \frac{\partial\phi_2}{\partial r} - \frac{\partial\phi_1}{\partial r}\Big|_{r=a} = \sum_{n=0}^{\infty} a^{n-1} \cdot n A_n P_n(\cos\theta) -$$

$$\sum_{n=0}^{\infty} -(n+1) a^{-(n+2)} B_n P_n(\cos\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(n a^{n-1} A_n + (n+1) a^{-(n+2)} B_n\right) \overset{*'}{=}$$

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\left(n a^{n-1} A_n + A_n a^{2n+1} (n+1) a^{-(n+2)}\right) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta)\left(n a^{n-1} + (n+1) a^{n-1}\right) =$$

$$\sum_{n=0}^{\infty} A_n P_n(\cos\theta)(2n+1) a^{n-1} = \overset{\sigma = \sigma_p}{\frac{P_0 \cos\theta}{\varepsilon_0}} \rightarrow A_n = 0 \quad n \in \mathbb{N} - \{1\}$$

$$\hookrightarrow A_1 \cdot 3 = \frac{P_0}{\varepsilon_0} \rightarrow A_1 = \frac{P_0}{3\varepsilon_0} \quad ; \quad B_n = 0 \quad n \in \mathbb{N} - \{1\} \quad B_1 = A_1 a^3 = \frac{P_0}{3\varepsilon_0} a^3$$

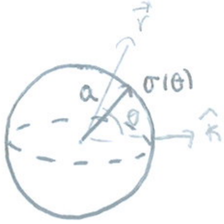
* Orduan

$$\phi(r,\theta) = \begin{cases} \dfrac{P_0}{3\varepsilon_0} r \cos\theta & r < a \\[2ex] \dfrac{P_0}{3\varepsilon_0} \dfrac{a^3}{r^2} \cos\theta & r \geq a \end{cases}$$

$$\vec{E}(r,\theta) = -\vec{\nabla}\phi(r,\theta) = \begin{cases} -\frac{P_0}{3\varepsilon_0}\cos\theta\,\hat{r} + \frac{P_0}{3\varepsilon_0}\sin\theta\,\hat{\theta} = -\frac{P_0}{3\varepsilon_0}\hat{k} = -\frac{\vec{P}_0}{3\varepsilon_0} & r<a \\ \frac{2P_0}{3\varepsilon_0}\frac{a^3}{r^3}\cos\theta\,\hat{r} + \frac{P_0}{3\varepsilon_0}\frac{a^3}{r^3}\sin\theta\,\hat{\theta} = \frac{P_0}{3\varepsilon_0}\left(\frac{a}{r}\right)^3(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}) & r\geq a \end{cases}$$

$$= -\left(\frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}\right)$$

1.10.)




* $\phi(\theta) = C\cos\theta = \phi(a,\theta)$ $C\in\mathbb{R}$

↳ $\sigma(\theta)$ gainazaleko karga dentsitateak sortua

Laplaceren ekuazioa → $\nabla^2\phi = 0$. $\varphi$-ren independentea →

$$\phi(r,\theta) = \sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+1)}\right]P_n(\cos\theta)$$

- $r\geq a$ → $A_n = 0$ $\forall n\in\mathbb{N}$, $\phi$ finitua delako $\forall r\in[0,\infty)$ (bestela $r\to\infty$ $\phi\to\infty$)

$$\phi_1(r,\theta) = \sum_{n=0}^{\infty} B_n \frac{1}{r^{n+1}} P_n(\cos\theta)$$

- $r<a$ → $B_n = 0$ $\forall n\in\mathbb{N}$, $\phi$ finitua delako $\forall r\in[0,\infty)$ (bestela $r\to 0$ $\phi\to\infty$)

$$\phi_2(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

* Mugalde baldintzak: $\phi$ jarraitua $r=a$ → $\phi_1(a,\theta) = \phi_2(a,\theta) = \phi(\theta) = C\cdot\cos\theta$ →

$$\phi_1(a,\theta) = \phi_2(a,\theta) \to B_n = A_n a^{2n+1}$$

$$\phi(a,\theta) = \sum_{n=0}^{\infty} A_n\cdot a^n P_n(\cos\theta) = C\cos\theta \to A_n = 0 \to n\in\mathbb{N}-\{1\}$$

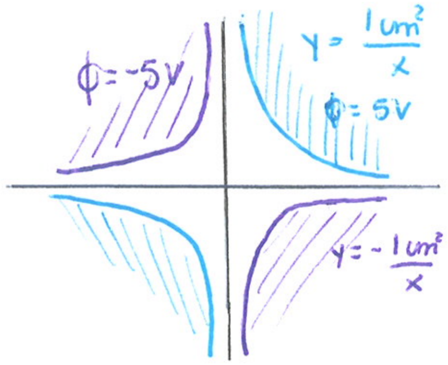
$$C = A_1 a \to A_1 = \frac{C}{a} \Leftrightarrow B_n = 0 \quad n\in\mathbb{N}-\{1\} \quad B_1 = A_1 a^3 = \frac{C}{a}\cdot a^3 = C\cdot a^2$$

Orduan → $$\phi(r,\theta) = \begin{cases} \frac{C}{a}\cdot r\cos\theta & r<a \\ C\frac{a^2}{r^2}\cos\theta & r\geq a \end{cases} \to \vec{E} = -\vec{\nabla}\phi = \begin{cases} -\frac{C}{a}\hat{k} & r<a \\ \frac{C}{a}\left(\frac{a}{r}\right)^3(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}) & r\geq a \end{cases}$$

* $$\left.\vec{E}_1 - \vec{E}_2\right|_{r=a}\cdot\hat{n} = \frac{\sigma(\theta)}{\varepsilon_0} = \left(\frac{C}{a}\cdot 2\cos\theta + \frac{C}{a}\cos\theta\right) = 3\frac{C}{a}\cos\theta \to \sigma(\theta) = \frac{3}{a}\varepsilon_0\cdot C\cos\theta$$

$\hat{n} = \hat{r}$

1.11)




$\phi(x, 1/x) = 5\,V \quad , \quad \phi(x, -1/x) = -5\,V$

a) $\phi(x,y) = Kxy \qquad K \in \mathbb{R}$ ?

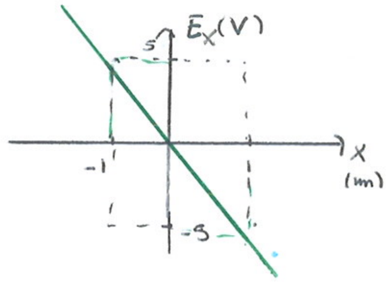
Laplace-n ekuazioa bete beharko du: $\nabla^2\phi = 0$

$\nabla^2\phi = \underbrace{\frac{\partial\phi^2}{\partial x^2} + \frac{\partial\phi^2}{\partial y^2}}_{\text{z-ren independentea}} = 0 + 0 = 0$ ✓ (Frogatzeko $\phi$ potentziala dela)

$\phi(x, 1/x) = K \cdot x \cdot \frac{1}{x} = K \cdot cm^2 = 5\,V \rightarrow K = 5\,V/cm^2$

(Gauza bera lortu $\phi(x, -1/x) = -5\,V$ eginez)

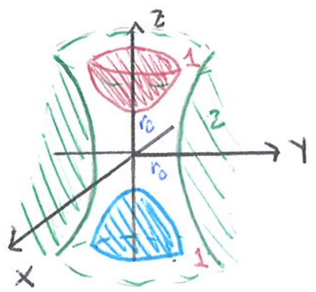
b) $\vec{E} = -\vec{\nabla}\phi = -\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}\right) = -K(y\hat{i} + x\hat{j})$




X ardatzean zehar $\rightarrow y = 0 \rightarrow \underbrace{\vec{E}(x,0)}_{\vec{E}_x} = -Kx\hat{j}$ (V/cm)

1.12)

Elektrodoen ekuazioak $\rightarrow$ 1: $2z^2 - x^2 - y^2 = 2r_0^2$ 2: $2z^2 - x^2 - y^2 = -r_0^2$




Z-rekiko simetrikoak

* 1: $2z^2 - x^2 - y^2 = 2r_0^2 \rightarrow 2z^2 - y^2 - 2r_0^2 = x^2 \rightarrow x = \pm\sqrt{2z^2 - y^2 - 2r_0^2}$

$\phi(\pm\sqrt{2z^2 - y^2 - 2r_0^2}, y, z) = 2V/3$

* 2: $2z^2 - x^2 - y^2 = -r_0^2 \rightarrow x = \pm\sqrt{2z^2 - y^2 + r_0^2}$

$\phi(\pm\sqrt{2z^2 - y^2 + r_0^2}, y, z) = -V/3$

Elektrodoen arteko potentziala $\rightarrow \phi(x,y,z) = \frac{V}{3r_0^2}(2z^2 - x^2 - y^2) \rightarrow$ Laplace-n ekuazioa bete beharko du $\rightarrow \nabla^2\phi = 0 \rightarrow$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{V}{3r_0^2}\left(-\frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(2y) + 2\frac{\partial}{\partial z}(2z)\right) = \frac{V}{3r_0^2}(-2-2+4) = 0 \checkmark$$

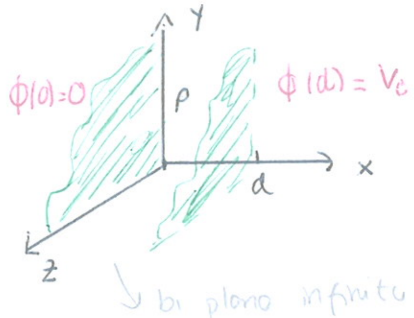
Gainera $\phi(\pm\sqrt{2z^2 - y^2 - 2r_0^2}, y, z) = \frac{2r_0^2 V}{3r_0^2} = 2V/3$ ✓

$\phi(\pm\sqrt{2z^2 - y^2 + r_0^2}, y, z) = -\frac{V r_0^2}{3r_0^2} = -V/3$ ✓

} Mugalde baldintzak betetzen dira

1.13.)

Poissonen ekuazioa → $\nabla^2\phi = -\rho/\varepsilon_0$ $(\rho \in \mathbb{R})$ M. B $\begin{cases} \phi(0,y,z)=0 \\ \phi(d,y,z)=V_0 \end{cases}$



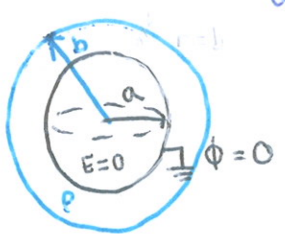

Bi plano infinitu →
y eta z-ren independentea potentziala

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} = \frac{\partial^2\phi}{\partial x^2} \rightarrow \frac{\partial\phi}{\partial x} = -\frac{\rho x}{\varepsilon_0} + A \rightarrow \phi(x) = -\frac{\rho x^2}{2\varepsilon_0} + Ax + B$$

Mugalde baldintzak → $\phi(0) = B = 0$ ; $\phi(d) = -\frac{\rho d^2}{2\varepsilon_0} + Ad = V_0 \rightarrow Ad = V_0 + \frac{\rho d^2}{2\varepsilon_0} \rightarrow$

$$A = \frac{V_0}{d} + \frac{\rho d}{2\varepsilon_0} \Rightarrow \phi(x) = -\frac{\rho x^2}{2\varepsilon_0} + \frac{V_0 x}{d} + \frac{\rho d x}{2\varepsilon_0} = \frac{\rho x}{2\varepsilon_0}(d-x) + \frac{V_0 x}{d}$$

1.14.)




a erradioko esfera eroalea. → lurrera konektatuta ; $\phi(a,\theta,\varphi)=0$

- $r<a$ → $\nabla^2\phi = 0$ , $\theta$ eta $\varphi$-ren independentea →

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0 \rightarrow r^2\frac{\partial\phi}{\partial r} = A \rightarrow \frac{\partial\phi}{\partial r} = \frac{A}{r^2} \rightarrow \phi = \frac{B}{r} + C$$

* $\phi(a) = \frac{B}{a} + C = 0 \rightarrow C = -B/a$

* Barruan ez dago kargarik beraz gai monopolarra 0 → $B=0$

(eroaleetan $E=0$) → $\phi(r)=0$

- $a \le r < b$ → Poissonen ekuazioa → $\nabla^2\phi = -\rho/\varepsilon_0$ $(\rho\in\mathbb{R})$ →

$$-\frac{\rho}{\varepsilon_0} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) \rightarrow \frac{\rho r^2}{\varepsilon_0} + \frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0 , \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = -\frac{\rho r^2}{\varepsilon_0} \rightarrow$$

$$r^2\frac{\partial\phi}{\partial r} = -\frac{\rho r^3}{3\varepsilon_0} + A \rightarrow \frac{\partial\phi}{\partial r} = -\frac{\rho r}{3\varepsilon_0} + \frac{A}{r^2} \rightarrow \phi(r) = -\frac{\rho r^2}{6\varepsilon_0} + \frac{B}{r} + C$$

* Mugalde baldintzak: $\phi(a) = -\frac{\rho a^2}{\varepsilon_0 6} + \frac{B}{a} + C = 0 \rightarrow C = \frac{\rho a^2}{6\varepsilon_0} - \frac{B}{a}$

$$\phi(r) = \frac{\rho}{6\varepsilon_0}(a^2-r^2) + B\left(\frac{1}{r} - \frac{1}{a}\right)$$

- $r \geq b$ → Laplacen ekuazioa → $\nabla^2\phi = 0 \rightarrow \phi(r) = \frac{D}{r} + E$ $\phi(\infty)=0 \rightarrow E=0$

Potentziala jarraitua → $\frac{\rho}{6\varepsilon_0}(a^2-b^2) + B\left(\frac{1}{b}-\frac{1}{a}\right) = \frac{D}{b} \rightarrow D = \frac{\rho b}{6\varepsilon_0}(a^2-b^2) + B\left(1-\frac{b}{a}\right)$

$Q = \oint_s \sigma \, ds = 0$ esfora neutra da

$\sigma = \varepsilon_0 \vec{E}_2 \cdot \hat{r} = \varepsilon_0 \left( \frac{-\partial \phi}{\partial r} \right) \Big|_{r=a} = \varepsilon_0 \left( \frac{\rho a}{3\varepsilon_0} + \frac{B}{a^2} \right) = \frac{\rho a}{3} + \frac{B}{a^2} \varepsilon_0$

$Q = \oint_s \sigma \, dS = \sigma \oint_s dS = \sigma \cdot 4\pi a^2 = 0 \longleftrightarrow \sigma = 0 \rightarrow \rho \frac{a}{3} = -\frac{B}{a^2} \varepsilon_0 \rightarrow B = -\frac{\rho a^3}{3\varepsilon_0}$

$\hookrightarrow \sigma = \text{cte}$

∘ $a < r < b$:

$\phi(r) = \frac{\rho}{6\varepsilon_0}(a^2 - r^2) - \frac{\rho a^3}{3\varepsilon_0}\left( \frac{1}{r} - \frac{1}{a} \right) = \frac{\rho a^2}{6\varepsilon_0} - \frac{\rho r^2}{6\varepsilon_0} - \frac{\rho a^3}{3\varepsilon_0 r} + \frac{\rho a^2}{3\varepsilon_0} = \frac{\rho a^2}{2\varepsilon_0} - \frac{\rho}{3\varepsilon_0}\left( \frac{r^2}{2} + \frac{a^3}{r} \right)$

∘ $r > b$

$D = \frac{\rho b}{6\varepsilon_0}(a^2 - b^2) + B\left(1 - \frac{b}{a}\right) = \frac{\rho b}{6\varepsilon_0}(a^2 - b^2) - \frac{\rho a^3}{3\varepsilon_0}\left(1 - \frac{b}{a}\right) = \frac{\rho b a^2}{6\varepsilon_0} - \frac{\rho b^3}{6\varepsilon_0} - \frac{\rho a^3}{3\varepsilon_0} + \frac{\rho b a^2}{3\varepsilon_0} =$

$\frac{\rho b a^2}{2\varepsilon_0} - \frac{\rho}{3\varepsilon_0}\left( \frac{b^3}{2} + a^3 \right) \Rightarrow \phi(r) = \frac{D}{r} = \frac{\rho b a^2}{2\varepsilon_0 r} - \frac{\rho}{3\varepsilon_0 r}\left( \frac{b^3}{2} + a^3 \right)$

* pdc $B = 0$ gai mangolora zero ioan behr duelelio

# ELEKTROMAGNETISMOA II

16-10-10

## MAGNETOSTATIKAKO MUGA-PROBLEMAK:

1.28.)

$\vec{B} = B_0 \hat{k}$


Banatuta kalkulatuko dugu potentzial eskalar magnetikoa gero eremu magnetikoa kalkulatzeko

$$\phi_M(r,\theta) = \begin{cases} \phi_1(r,\theta) & r < a \\ \phi_2(r,\theta) & r \geq a \end{cases}$$

↳ Simetriagatik $\varphi$-ren independentea.

1 ⇒ Lehenengo $\phi_1(r,\theta)$ kalkulatuko dugu: Laplace ; $\nabla^2 \phi_M = 0$ →

$$\phi_1(r,\theta) = \sum_{n=0}^{\infty} [A_n r^n + B_n r^{-(n+1)}] P_n(\cos\theta)$$

Zentruan potentziala finitua izan behar denez $B_n = 0$ izan beharko da $\forall n \in \mathbb{N}$. ⇒ $\phi_1(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \, P_n(\cos\theta)$

2 ⇒ $\phi_2(r,\theta)$ kalkulatuko dugu: Laplace ; $\nabla^2 \phi_M = 0$ →

$$\phi_2(r,\theta) = \sum_{n=0}^{\infty} [A_n' r^n + B_n' r^{-(n+1)}] P_n(\cos\theta)$$

$$\vec{B}(r\to\infty,\theta) = B_0 \hat{k} \;;\; \vec{H}(r\to\infty,\theta) = \frac{B_0}{\mu_0}\hat{k} = -\vec{\nabla}\phi_M \to \phi_2(r,\theta)_{r\to\infty} = -\frac{B_0}{\mu_0} r\cos\theta$$

$$\phi_2(r\to\infty,\theta) = \sum_{n=0}^{\infty} A_n' r^n \, P_n(\cos\theta) = -\frac{B_0}{\mu_0} r\cos\theta \to A_0' = 0 \,,\; A_1' = -\frac{B_0}{\mu_0}$$

$$A_n' = 0 \quad \forall n \in \mathbb{N} - \{1\} \Rightarrow \phi_2(r,\theta) = -\frac{B_0}{\mu_0} r\cos\theta + \sum_{n=0}^{\infty} B_n' r^{-(n+1)} P_n(\cos\theta)$$

$H_{1t} = H_{2t}$

* Potentziala jarraitua izan behar da; $\phi_2(a,\theta) = \phi_1(a,\theta)$ izan beharko da.

$$\phi_1(a,\theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) = \phi_2(a,\theta) = -\frac{B_0}{\mu_0} a\cos\theta + \sum_{n=0}^{\infty} B_n' a^{-(n+1)} P_n(\cos\theta)$$

$$\Rightarrow \quad \frac{B_0}{\mu_0} a\cos\theta = \sum_{n=0}^{\infty}\left[B_n' a^{-(n+1)} - A_n a^n\right] P_n(\cos\theta) \longleftrightarrow \quad B_n' a^{-(n+1)} - A_n a^n = 0 \quad \forall n\in\mathbb{N}-\{1\}$$

*

$$\frac{B_1'}{a^2} - A_1 a = \frac{B_0}{\mu_0} a \rightarrow \frac{B_1'}{a^2} = a\left(A_1 + \frac{B_0}{\mu_0}\right) \rightarrow B_1' = a^3\left(A_1 + \frac{B_0}{\mu_0}\right) \quad (1)$$

$$* \quad \frac{B_n'}{a^{n+1}} = A_n a^n \rightarrow B_n' = A_n a^{2n+1} \quad \forall n\in\mathbb{N}-\{1\} \quad (2)$$

$$* \quad B_{1n} = B_{2n} \rightarrow \quad -\mu\left[\frac{\partial\phi_1}{\partial r}\right]_{r=a} = -\mu_0\left[\frac{\partial\phi_2}{\partial r}\right]_{r=a} \Rightarrow$$

r=a denean

$$-\mu\sum_{n=0}^{\infty} n A_n a^{n-1} P_n(\cos\theta) = -\mu_0\left(-\frac{B_0}{\mu_0}\cos\theta + \sum_{n=0}^{\infty} -(n+1) B_n' a^{-(n+2)} P_n(\cos\theta)\right) =$$

$$B_0\cos\theta + \mu_0\sum_{n=0}^{\infty}(n+1)\frac{B_n'}{a^{n+2}} P_n(\cos\theta) \Rightarrow \quad -B_0\cos\theta = \sum_{n=0}^{\infty}\left[\mu_0(n+1)\frac{B_n'}{a^{n+2}} + \mu n A_n a^{n-1}\right] P_n(\cos\theta)$$

$$\bullet \quad -B_0 = \mu_0\cdot 2\frac{B_1'}{a^3} + \mu A_1 \rightarrow -\mu A_1 - B_0 = \frac{2\mu_0}{a^3} B_1' \overset{(1)}{=} \frac{2\mu_0}{\cancel{a^3}}\cancel{a^3}\left(A_1 + \frac{B_0}{\mu_0}\right) = 2\mu_0 A_1 + 2B_0 \rightarrow$$

$$-A_1(\mu + 2\mu_0) = 3B_0 \rightarrow \quad A_1 = \frac{-3B_0}{\mu+2\mu_0} \quad ; \quad B_1' = a^3\left(-\frac{3B_0}{\mu+2\mu_0} + \frac{B_0}{\mu_0}\right) =$$

$$a^3\left(\frac{-3B_0\mu_0 + B_0\mu + 2B_0\mu_0}{\mu_0(\mu+2\mu_0)}\right) = a^3 B_0\frac{(\mu-\mu_0)}{\mu_0(\mu+2\mu_0)} = \left(\frac{\mu}{\mu_0} - 1\right)\frac{B_0 a^3}{\mu+2\mu_0}$$

$$\bullet \quad \mu_0(n+1)\frac{B_n'}{a^{n+2}} + \mu n A_n a^{n-1} = 0 \rightarrow \mu_0(n+1) B_n' = -\mu n A_n a^{2n+1} \rightarrow B_n' = \frac{-\mu\cdot n}{\mu_0(n+1)} A_n a^{2n+1} \quad (3)$$

$\forall n\in\mathbb{N}-\{1\}$

(3) eta (2) berdinduz $\quad B_n' = \frac{-\mu n}{\mu_0(n+1)}\cancel{A_n} a^{2n+1} = A_n a^{2n+1} \rightarrow \frac{-\mu n}{\mu_0(n+1)} = 1 \rightarrow -\mu n \cancel{=} \mu_0(n+1)$

ezinezkoa, $\mu, \mu_0, n \geq 0$

Beraz aldera baliorra biak beteko lirateke ((2) eta (3)) $A_n$ eta $B_n'$ direlako

zero izatea da $\rightarrow A_n = B_n' = 0 \quad \forall n\in\mathbb{N}-\{1\}$

Hortaz $\quad \phi_M(r,\theta) = \begin{cases} \phi_1(r,\theta) = \dfrac{-3B_0 r\cos\theta}{\mu+2\mu_0} & r<a \\[2ex] \phi_2(r,\theta) = -\dfrac{B_0 r\cos\theta}{\mu_0} + \left(\dfrac{\mu}{\mu_0}-1\right) a^3\dfrac{B_0}{2\mu_0+\mu}\cdot\dfrac{\cos\theta}{r^2} & r\geq a \end{cases}$

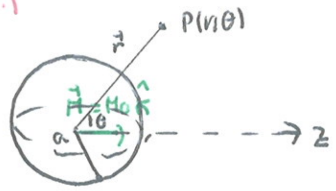
$$\vec{H} = -\vec{\nabla}\phi_M = \begin{cases} +\frac{3B_0\cos\theta}{\mu+2\mu_0}\hat{r} - \frac{3B_0}{\mu+2\mu_0}\sin\theta\,\hat{\theta} = \frac{3B_0}{\mu+2\mu_0}\hat{k} & r<a \\ +\frac{B_0}{\mu_0}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) + \left(\frac{\mu}{\mu_0}-1\right)a^3\frac{B_0}{2\mu_0+\mu}\left(\frac{\sin\theta}{r^3}\hat{\theta} + \frac{2\cos\theta}{r^3}\hat{r}\right) = \frac{B_0}{\mu_0}\hat{k} + \left(\frac{\mu}{\mu_0}-1\right)\left(\frac{a}{r}\right)^3\frac{B_0}{2\mu_0+\mu}(\sin\theta\hat{\theta}+2\cos\theta\hat{r}) & r\geq a \end{cases}$$

$$\vec{B} = \vec{H}\mu = \begin{cases} \frac{3B_0\mu}{\mu+2\mu_0}\hat{k} & r<a \\ B_0\hat{k} + \left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)\left(\frac{a}{r}\right)^3 B_0(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) & r\geq a \end{cases}$$

$$\vec{M} = \frac{\vec{B}_1}{\mu_0} - \vec{H}_1 = \frac{3B_0}{\mu_0}\left(\frac{\mu}{\mu+2\mu_0}\right)\hat{k} - \frac{3B_0}{\mu+2\mu_0}\hat{k} = \frac{3B_0}{\mu+2\mu_0}\left(\frac{\mu}{\mu_0}-1\right)\hat{k} = \frac{3B_0}{\mu_0}\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)\hat{k}$$

$$\vec{B}_1 = \mu_0(\vec{M}+\vec{H}_1)$$

1.29.)




Simetriagatik eremua eta potentzial eskalar magnetikoa $\varphi$-ren independenteak izango dira.

Banatuta kalkulatuko dugu alde batetik barneko potentzial eskalar magnetikoa eta bestetik kanpokoa.

$$\phi_M(r,\theta) = \begin{cases} \phi_1(r,\theta) & r<a \\ \phi_2(r,\theta) & r\geq a \end{cases}$$

1 ⇒ Laplace → $\nabla^2\phi_M = 0$ → $\phi_M = \sum_{n=0}^{\infty}[A_n r^n + B_n r^{-(n+1)}]P_n(\cos\theta)$

Zentruan potentziala finitua denez, $B_n = 0 \quad \forall n\in\mathbb{N} \Rightarrow \phi_1 = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$

2 ⇒ Laplace → $\nabla^2\phi_M = 0$ → $\phi_M = \sum_{n=0}^{\infty}[C_n r^n + D_n r^{-(n+1)}]P_n(\cos\theta)$

Infinituan, ($\forall r$ ordenean), finitua denez $C_n = 0 \quad \forall n\in\mathbb{N} \to \phi_2 = \sum_{n=0}^{\infty} D_n r^{-(n+1)} P_n(\cos\theta)$

- Mugalde baldintzak → $H_{1t} = H_{2t}$ ; $r=a$ puntuan potentziala jarraitua izan behar da

$$\phi_1(a,\theta) = \phi_2(a,\theta) \leftrightarrow \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) = \sum_{n=0}^{\infty} D_n a^{-(n+1)} P_n(\cos\theta) \Rightarrow$$

$$A_n a^n = D_n a^{-(n+1)} \to D_n = A_n a^{2n+1} \quad \forall n\in\mathbb{N} \quad (1)$$

- $B_{1n} = B_{2n}\big|_{r=a}$ → $\vec{B}_{1n} = \mu_0(\vec{H}_{1n} + \vec{M}_r) = \mu_0\left(-\frac{\partial\phi_1}{\partial r}\hat{r} + \vec{M}_r\right)$ ; $\vec{B}_{2n} = \mu_0\vec{H}_{2n} = -\mu_0\frac{\partial\phi_2}{\partial r}\hat{r}$

$$-\mu_0 \left(\frac{\partial \Phi_2}{\partial r}\right)\Bigg|_{r=a} = -\mu_0 \sum_{n=0}^{\infty} -(n+1) D_n\, a^{-(n+2)} P_n(\cos\theta) \underset{(1)}{=} \sum_{n=0}^{\infty} \mu_0 (n+1) A_n \frac{a^{2n+1}}{a^{n+2}} P_n(\cos\theta) =$$

$$\sum_{n=0}^{\infty} \mu_0(n+1) A_n a^{n-1} P_n(\cos\theta) = -\mu_0 \left[\left(\frac{\partial \Phi_1}{\partial r}\right) - M_0 \cos\theta\right]\Bigg|_{r=a} = \mu_0 \left[-\sum_{n=0}^{\infty} n A_n a^{n-1} P_n(\cos\theta) + M_0 \cos\theta\right] =$$

$$\sum_{n=0}^{\infty} -\mu_0 n A_n a^{n-1} P_n(\cos\theta) + M_0 \mu_0 \cos\theta \Rightarrow \sum_{n=0}^{\infty} \left[\mu_0(n+1) + \mu_0 n\right] A_n a^{n-1} P_n(\cos\theta) = M_0 \mu_0 \cos\theta$$

$A_n = 0 \quad \forall n \in \mathbb{N} - \{1\}$ $\qquad A_1 = \frac{M_0 \cancel{\mu_0}}{3\cancel{\mu_0}} = \frac{M_0}{3}$ ; $D_1 = A_1 a^3 = \frac{M_0}{3} a^3$

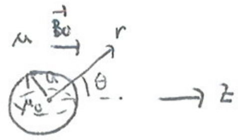
↳ $D_n = 0 \quad \forall n \in \mathbb{N} - \{1\}$

Berat $\Rightarrow$ $\Phi_M(r,\theta) = \begin{cases} \frac{M_0}{3} r \cos\theta & r < a \\ \frac{M_0}{3} \frac{a^3}{r^2} \cos\theta & r \geqslant 0 \end{cases}$ $\Rightarrow$ $\vec{H} = -\vec{\nabla}\Phi_M = -\frac{\partial \Phi_M}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \Phi_M}{\partial \theta}\hat{\theta}$

$$\vec{H} = \begin{cases} -\frac{M_0}{3}\cos\theta\,\hat{r} + \frac{M_0}{3}\sin\theta\,\hat{\theta} = -\frac{M_0}{3}\hat{k} = -\frac{\vec{M}}{3} & r < a \\ \frac{M_0}{3}\frac{a^3}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) & r \geqslant a \end{cases} \Rightarrow$$

$$\vec{B} = \mu\vec{H} \underset{\mu_0(\vec{H}+\vec{M})}{=} \begin{cases} -\frac{\vec{M}}{3}\mu_0 + \vec{M}\mu_0 = \frac{2}{3}\mu_0 \vec{M} & r \leqslant a \\ \frac{M_0 \mu_0}{3}\left(\frac{a}{r}\right)^3 (2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) & r \geqslant a \end{cases}$$

1.31.)




$\vec{B}_0 = B_0 \hat{k}$

$a \ll$ materialaren dimentsioak $\longleftrightarrow$ 1.28 ariketaren berdina baina alderantziz?

Bananta kalkulatuko ditugu potentzial eskalar magnetikoa materialean eta zulo esferikoan. $\Phi_M(r,\theta) = \begin{cases} \Phi_1(r,\theta) & r < a \\ \Phi_2(r,\theta) & r \geqslant a \end{cases}$

1 $\Rightarrow$ $\nabla^2 \Phi_M = 0 \rightarrow \Phi_M = \Phi_1(r,\theta) = \sum_{n=0}^{\infty} \left[A_n r^n + B_n r^{-(n+1)}\right] P_n(\cos\theta)$

Zentruan potentziala finitua izan behar denez $B_n = 0$ izan behar da $\rightarrow$

$$\Phi_M = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

$2 \Rightarrow \quad \nabla^2 \phi_M = 0 \rightarrow \phi_M = \phi_2(r,\theta) = \sum_{n=0}^{\infty} [C_n r^n + D_n r^{-(n+1)}] P_n(\cos\theta)$

$r \to \infty$ denean $\vec{B}(r\to\infty,\theta) = B_0 \hat{k} \rightarrow \phi_2(r\to\infty,\theta) = -\frac{B_0}{\mu} r\cos\theta$

$\hookrightarrow (r \gg a)$

$$\phi_2(r\to\infty,\theta) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta) = -\frac{B_0}{\mu} r\cos\theta \leftrightarrow C_1 = -\frac{B_0}{\mu} \qquad C_n = 0 \quad \forall n \in \mathbb{N}-\{1\}$$

$$\phi_2(r,\theta) = \sum_{n=0}^{\infty} D_n r^{-(n+1)} P_n(\cos\theta) - \frac{B_0}{\mu} r\cos\theta$$

* Potentziala jarraitua izan behar da $r=a$-n $\rightarrow \phi_2(a,\theta) = \phi_1(a,\theta)$:

$$\phi_2(a,\theta) = \sum_{n=0}^{\infty} D_n \frac{1}{a^{n+1}} P_n(\cos\theta) - \frac{B_0}{\mu} a\cos\theta = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) \rightarrow$$

$$\sum_{n=0}^{\infty} [D_n a^{-(n+1)} - A_n a^n] P_n(\cos\theta) = \frac{B_0}{\mu} a\cos\theta \Rightarrow D_1 a^{-2} - A_1 a = \frac{B_0}{\mu} a \rightarrow$$

$$\frac{D_1}{a^2} = a\left(A_1 + \frac{B_0}{\mu}\right) \rightarrow D_1 = a^3\left(A_1 + \frac{B_0}{\mu}\right) ; \quad D_n a^{-(n+1)} - A_n a^n = 0 \quad \forall n \in \mathbb{N}-\{1\}$$

$$D_n = A_n a^{2n+1} \quad \forall n \in \mathbb{N}-\{1\} \qquad (1)$$

* $\underset{r=a}{B_{1n} = B_{2n}} \rightarrow \mu_0 \left[-\frac{\partial \phi_1}{\partial r}\right]_{r=a} = \mu \cdot \left[-\frac{\partial \phi_2}{\partial r}\right]_{r=a} \Rightarrow \mu_0 \left(-\sum_{n=0}^{\infty} n A_n a^{n-1} P_n(\cos\theta)\right) =$

$$\mu\left(-\sum_{n=0}^{\infty} -(n+1) D_n a^{-(n+2)} P_n(\cos\theta) + \frac{B_0}{\mu}\cos\theta\right) = B_0\cos\theta + \sum_{n=0}^{\infty} \mu \frac{(n+1)D_n}{a^{n+2}} P_n(\cos\theta) =$$

$$\sum_{n=0}^{\infty} -\mu_0 n A_n a^{n-1} P_n(\cos\theta) \rightarrow -B_0\cos\theta = \sum_{n=0}^{\infty}\left[\frac{\mu(n+1)D_n}{a^{n+2}} + \mu_0 n A_n a^{n-1}\right] P_n(\cos\theta)$$

$$\Rightarrow -B_0 = \frac{2\mu}{a^3} D_1 + \mu_0 A_1 \rightarrow -\frac{B_0}{\mu_0} - \frac{2\mu D_1}{a^3 \mu_0} = A_1 \rightarrow D_1 = a^3\left(-\frac{B_0}{\mu_0} - \frac{2\mu D_1}{a^3\mu_0} + \frac{B_0}{\mu}\right) =$$

$$a^3 B_0 \left(\frac{1}{\mu} - \frac{1}{\mu_0}\right) - 2\frac{\mu}{\mu_0} D_1 \rightarrow D_1\left(1 + 2\frac{\mu}{\mu_0}\right) = a^3 B_0 \left(\frac{1}{\mu} - \frac{1}{\mu_0}\right) = a^3 B_0 \left(\frac{\mu_0 - \mu}{\mu \cdot \cancel{\mu_0}}\right) =$$

$$D_1\left(\frac{\mu_0 + 2\mu}{\cancel{\mu_0}}\right) \rightarrow D_1 = a^3 \frac{B_0}{\mu}\left(\frac{\mu_0 - \mu}{\mu_0 + 2\mu}\right) = \frac{a^3 B_0}{\mu_0 + 2\mu}\left(\frac{\mu_0}{\mu} - 1\right) ; \quad A_1 = -\frac{3 B_0}{\mu_0 + 2\mu}$$

$$\Rightarrow \quad \frac{\mu(n+1) D_n}{a^{n+2}} + \mu_0 n A_n a^{n-1} = 0 \quad (2) \quad \rightarrow \quad -\frac{\mu(n+1)}{a^{n+2}} \frac{D_n}{\mu_0 n a^{n-1}} = A_n$$

(1) eta (2) berdinduz

$$(1) \quad A_n = -\frac{\mu(n+1) D_n}{n \mu_0} \cdot \frac{1}{a^{2n+1}} \quad ; \quad (2) \quad D_n = A_n a^{2n+1} = -\frac{\mu(n+1) D_n}{n \mu_0} \rightarrow D_n \left(1 + \frac{\mu(n+1)}{n\mu_0}\right) = 0$$

$$1 + \frac{\mu(n+1)}{n\mu_0} = 0 \rightarrow \cancel{\frac{\mu(n+1)}{n\mu_0} = -1} \text{ ezinezkoa } \frac{\mu(n+1)}{n\mu_0} \geq 0 \text{ delako} \iff D_n = A_n = 0$$

$$\forall n \in \mathbb{N} - \{1\}$$

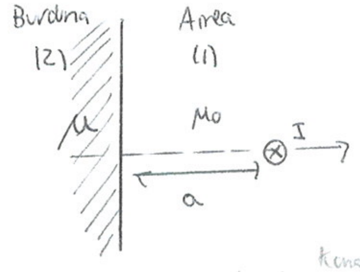
Beraz,

$$\phi_M(r,\theta) = \begin{cases} \phi_1(r,\theta) = \dfrac{-3B_0}{\mu_0 + 2\mu} r\cos\theta & r < a \\ \phi_2(r,\theta) = -\dfrac{B_0}{\mu} r\cos\theta + \left(\dfrac{\mu_0}{\mu} - 1\right) a^3 \dfrac{B_0}{2\mu + \mu_0} \dfrac{\cos\theta}{r^2} & r \geq a \end{cases} \Rightarrow$$

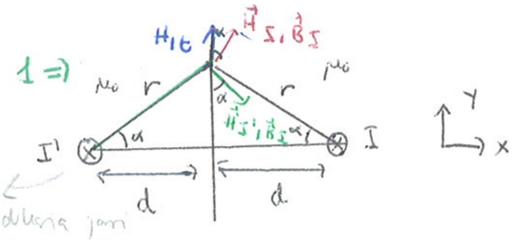
$$\vec{H} = -\vec{\nabla}\phi_M = \begin{cases} \dfrac{3B_0}{\mu_0 + 2\mu} \hat{k} & r < a \\ \dfrac{B_0}{\mu} \hat{k} + \left(\dfrac{\mu_0}{\mu} - 1\right)\left(\dfrac{a}{r}\right)^3 \dfrac{B_0}{2\mu + \mu_0} (\sin\theta\, \hat{\theta} + 2\cos\theta\, \hat{r}) & r \geq a \end{cases}$$

$$\vec{B} = \begin{cases} \dfrac{3B_0 \mu_0}{\mu_0 + 2\mu} \hat{k} & r < a \\ B_0 \hat{k} + \left(\dfrac{\mu_0 - \mu}{2\mu + \mu_0}\right)\left(\dfrac{a}{r}\right)^3 B_0 (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta}) & r \geq a \end{cases}$$

1.32)




Muga baldintzak: $\begin{cases} B_{1n} = B_{2n} \\ H_{1t} = H_{2t} \end{cases}$ ; Material magnetikoetako korronte induzituen metodoa $\Rightarrow$




korronte induzitua jarri aztertzen dugun beste aldean (norantza bera eremua 1. gunean egon dadin)

$$\begin{cases} H_I(r) = \dfrac{I}{2\pi r} & (\text{Ampère}) \\ H_{I_t}(r) = H_I \cos\alpha = H_I \dfrac{d}{r} = \dfrac{I d}{2\pi r^2} \end{cases}$$

$$H_{I'}(r) = \frac{I'}{2\pi r} \quad , \quad H_{I't}(r) = H_{I'} \cos\alpha = \frac{I' d}{2\pi r^2}$$

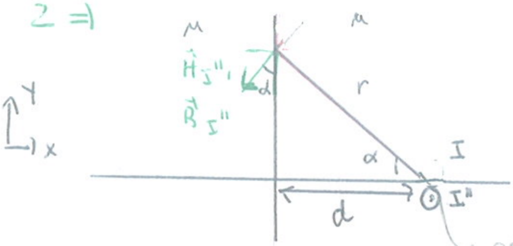
$$* \quad \vec{H}_{1t} = \vec{H}_{It} + \vec{H}_{I't} = \frac{d}{2\pi r^2}(I - I')\,\hat{j} \qquad |\vec{H}_{1t}| = \frac{d}{2\pi r^2}|I - I'|$$

$$B_I(r)=\frac{I}{2\pi r}\mu_0 \quad , \quad B_{I'}(r)=\frac{I'}{2\pi r}\mu_0 \quad ; \quad B_{In}(r)=\frac{I}{2\pi r}\mu_0 \sin\alpha=\frac{I}{2\pi r}\mu_0\frac{y}{r}=\frac{I\mu_0 y}{2\pi r^2}$$

$$B_{I'n}(r)=\frac{I'}{2\pi r}\mu_0\sin\alpha=\frac{I'}{2\pi r}\mu_0\frac{y}{r}=\frac{I'\mu_0 y}{2\pi r^2}$$

$$* \quad \vec B_{1n}=\vec B_{In}+\vec B_{I'n}=\frac{\mu_0 y}{2\pi r^2}(I+I')\,\vec i \qquad |\vec B_{1n}|=\frac{\mu_0 y}{2\pi r^2}(I+I')$$

2 ⇒




positivo baten, baina aurkako noranzkoan; bakarra jarri, jar genitzake biak eta bien eremua kalkulatu

$$\begin{cases} \cancel{H_I(r)=\frac{I}{2\pi r}} \quad , \quad \cancel{H_{It}(r)=\frac{I}{2\pi r}\cos\alpha=\frac{I}{2\pi r}\frac{d}{r}=\frac{Id}{2\pi r^2}} \\ H_{I''}(r)=\frac{I''}{2\pi r} \quad , \quad H_{I''t}(r)=\frac{I''}{2\pi r}\cos\alpha=\frac{I''d}{2\pi r^2}\end{cases}$$

$$B_{I''}(r)=\mu H_{I''}(r)=\frac{\mu I''}{2\pi r} \quad ; \quad B_{I''n}(r)=\frac{\mu I''}{2\pi r}\sin\alpha=\mu\frac{I''y}{2\pi r^2}$$

$$* \quad \vec B_{2n}=\vec B_{I''n}=-\frac{\mu I'' y}{2\pi r^2}\vec i \quad ; \quad \vec H_{2t}=\vec H_{I''t}=-\frac{d}{2\pi r^2}I''\,\hat j$$

- Mugalde baldintzak aplikatu:

$$* \quad \vec B_{1n}-\vec B_{2n}=0 \rightarrow \mu\frac{I''\cancel{y}}{\cancel{2\pi r^2}}\vec i+\frac{\mu_0\cancel{y}}{\cancel{2\pi r^2}}(I+I')\vec i=0 \leftrightarrow \mu I''=-\mu_0(I+I')$$

$$I''=-\frac{\mu_0}{\mu}(I+I')$$

$$* \quad \vec H_{1t}-\vec H_{2t}=0 \rightarrow \frac{\cancel{d}}{\cancel{2\pi r^2}}(I-I')\hat j+\frac{\cancel{d}}{\cancel{2\pi r^2}}I''\hat j=0 \leftrightarrow -I+I'=I''$$

$$I''=-I+I'=-\frac{\mu_0}{\mu}(I+I') \rightarrow I-I'=+\frac{\mu_0}{\mu}I+\frac{\mu_0}{\mu}I' \rightarrow$$

$$I'\left(\frac{\mu_0}{\mu}+1\right)=I\left(1-\frac{\mu_0}{\mu}\right)=I\left(\frac{\mu-\mu_0}{\cancel{\mu}}\right)=I'\left(\frac{\mu_0+\mu}{\cancel{\mu}}\right) \Rightarrow I'=I\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)$$

$$I''=I\left(\frac{\mu-\mu_0}{\mu+\mu_0}-1\right)=I\left(\frac{\cancel{\mu}-\mu_0-\cancel{\mu}-\mu_0}{\mu+\mu_0}\right)=\frac{-2\mu_0}{\mu+\mu_0}I \quad \text{(aurkako noranzkoan)}$$

Beraz, 1. eremuan $\vec B_1=\vec B_{1I}+\vec B_{1I'}=\left(\frac{I}{2\pi r}\mu_0(\cos\alpha\,\hat j+\sin\alpha\,\vec i)+\frac{I'}{2\pi r}\mu_0(-\cos\alpha\,\hat j+\sin\alpha\,\vec i)\right)=$

$$\frac{\mu_0}{2\pi r}\left(I\cos\alpha\,\hat j+I\sin\alpha\,\hat j-I\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\cos\alpha\,\hat j+I\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\sin\alpha\,\vec i\right)=\frac{I\mu_0}{2\pi r}\left(\cos\alpha\left(1-\frac{\mu-\mu_0}{\mu+\mu_0}\right)\hat j+\right.$$

$$\left.\sin\alpha\left(1+\frac{\mu-\mu_0}{\mu+\mu_0}\right)\vec i\right)=\frac{I\mu_0}{2\pi r}\left(\cos\alpha\left(\frac{2\mu_0}{\mu+\mu_0}\right)\hat j+\sin\alpha\left(\frac{2\mu}{\mu+\mu_0}\right)\vec i\right)=\frac{\cancel{2}I\mu_0}{\cancel{2}\pi r^2(\mu+\mu_0)}\left((d-x)\mu_0\hat j+y\,\mu\vec i\right)$$

$$\sin\alpha=y/r$$
$$\cos\alpha=(d-x)/r$$

2. eremuon: $\vec{B}_2 = \vec{B}_{2I''} = -\frac{I''\mu}{2\pi r}(+\sin\alpha\,\vec{i} + \cos\alpha\,\vec{j}) = \frac{2I\mu_0\mu}{2\pi r(\mu+\mu_0)}(\sin\alpha\,\vec{i} + \cos\alpha\,\vec{j}) =$

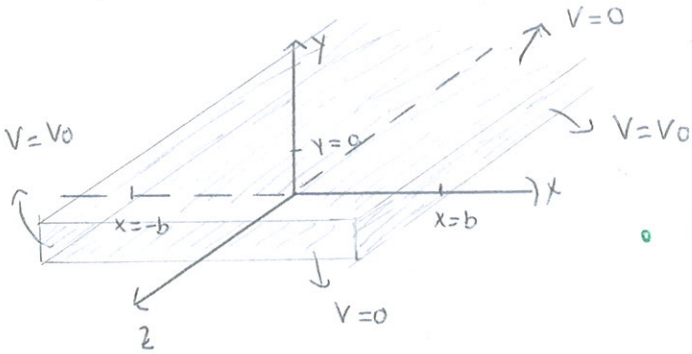
$$I'' = \frac{-2\mu_0}{\mu+\mu_0} I$$

$$\frac{I\mu_0\mu}{\pi r^2(\mu+\mu_0)}\left( y\,\vec{i} + (d-x)\,\vec{j}\right)$$

# ELEKTROMAGNETISMOA II AZTERKETAK:

## 2013-2014    1. Partziala:

1.) Demagun lurrera lotuta dauden bi XZ mugagabeko planoak ditugula, $y=0$ eta $y=a$ posizioetan kokatuta daudenak. Espazioa mugatuz, beste bi plano dauzkagu $x=\pm b$ posizioetan, baina plano horien potentziala $V_0$ (balioitzerena) delarik. Lor ezazu potentzialaren balioa (adierazpena) plano lau horien arteko espazioko edozein puntuan.




- Simetriagatik potentziala, $\phi$, z-ren independentea izango da (z norabidean infinitua da) $\Rightarrow \phi = \phi(x,y)$

- Mugalde baldintzak:

$$\phi(b,y) = V_0 \quad , \quad \phi(-b,y) = V_0 \qquad (0<y<a)$$

$$\phi(x,0) = 0 \quad , \quad \phi(x,a) = 0 \qquad (-b<x<b)$$

- Plano lauen arteko espazioan ez dago kargarik $\Rightarrow \rho = 0 \Rightarrow \nabla^2\phi = 0$ (Laplace-n ekuazioa):

$$\phi = \phi(x,y) \Rightarrow \quad \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

; Aldagaien banantzea: $\phi(x,y) = X(x)\,Y(y) \Rightarrow$

$$\nabla^2\phi = YX'' + X\ddot{Y} = 0 \iff \frac{X''}{X} + \frac{\ddot{Y}}{Y} = 0 \iff \frac{X''}{X} = -\frac{\ddot{Y}}{Y} = K$$

kte bat.

↳ $\left[\ ' \equiv \frac{d}{dx} \ , \ \dot{} \equiv \frac{d}{dy}\right]$

- $\phi(x,0) = X(x)Y(0) = 0 \iff Y(0) = 0$ ; $\phi(x,a) = X(x)Y(a) = 0 \iff Y(a) = 0$

- $K=0 \Rightarrow \frac{\ddot{Y}}{Y} = 0 \rightarrow \ddot{Y} = 0 \rightarrow Y = Ay + B$ ; $Y(0) = B = 0$ , $Y(a) = A\cdot a = 0 \rightarrow A = 0$

  Soluzio tribiala $\Rightarrow K \neq 0$

- $K<0 \Rightarrow \frac{\ddot{Y}}{Y} = +|K| = -K \rightarrow \ddot{Y} + KY = 0 \rightarrow \ddot{Y} - |K|Y = 0 \rightarrow Y = Ae^{\sqrt{|K|}y} + Be^{-\sqrt{|K|}y}$

$Y(0) = A + B = 0 \rightarrow A = -B \leftrightarrow Y(y) = A\left(e^{\sqrt{|k|}\,y} - e^{-\sqrt{|k|}\,y}\right) = \frac{A}{2} \sinh \sqrt{|k|}\,y$

$Y(a) = \frac{A}{2} \sinh \sqrt{|k|}\,a = 0 \rightarrow e^{\sqrt{|k|}\,a} - e^{-\sqrt{|k|}\,a} = 0 \rightarrow e^{2\sqrt{|k|}\,a} = 1 \Rightarrow$

ezinezkoa $\overset{\text{enecla!}}{|k| \neq 0} \leftrightarrow A = 0 \Rightarrow Y = 0$ soluzio tribiala $\leftrightarrow$ beraz $k > 0$

- $k > 0 \Rightarrow \frac{\ddot{Y}}{Y} = -k \rightarrow \ddot{Y} + kY = 0 \rightarrow Y = A \sin \sqrt{k}\,y + B \cos \sqrt{k}\,y$

$Y(0) = B = 0 \rightarrow Y(y) = A \sin \sqrt{k}\,y$ ; $Y(a) = A \sin \sqrt{k}\,a = 0 \rightarrow \sqrt{k}\,a = n\pi \quad n \in \mathbb{N} \rightarrow$

$k = \left(\frac{n\pi}{a}\right)^2 \quad n \in \mathbb{N} - \{0\}$ ; $Y(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} y\right)$ $\left\{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} y\right)\right\}$ oinarri ortonormala

Beraz $\Rightarrow \frac{\ddot{X}}{X} = k \rightarrow \ddot{X} - kX = 0 \rightarrow X(x) = Ce^{\sqrt{k}\,x} + De^{-\sqrt{k}\,x}$

* $\phi(x,y) = X(x)Y(y) = \sum_n A \sin \sqrt{k}\,y \left(Ce^{\sqrt{k}\,x} + De^{-\sqrt{k}\,x}\right) = \sum_n \left(\tilde{C}e^{\sqrt{k}\,x} + \tilde{D}e^{-\sqrt{k}\,x}\right) \sin \sqrt{k}\,y$

$\left(k = \left(\frac{n\pi}{a}\right)^2 \quad n \in \mathbb{N} - \{0\}\right) \Rightarrow \phi(x,y) = \sum_{n=1}^{\infty} \left(C_n e^{\frac{n\pi}{a}x} + D_n e^{-\frac{n\pi}{a}x}\right) \sin \frac{n\pi}{a} y$

- $\phi(b,y) = \sum_{n=1}^{\infty} \left(C_n e^{\frac{n\pi}{a}b} + D_n e^{-\frac{n\pi}{a}b}\right) \sin \frac{n\pi}{a} y = V_0 \underset{*}{=} \sum_{n=1}^{\infty} V_0 \frac{a}{n\pi} \left((-1)^n - 1\right) \sin \frac{n\pi}{a} y$

* Garatu $V_0$ $\left\{\sin \frac{n\pi}{a} y\right\}_{n=1}^{\infty}$ oinarrian $\Rightarrow V_0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{a} y \rightarrow$

$a_n = \underbrace{\frac{2}{a}}_{\text{norma karratuagatik zatitu}}\left(V_0, \sin \frac{n\pi}{a} y\right) = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n\pi}{a} y\right) dy = -V_0 \frac{2}{n\pi} \cos \frac{n\pi}{a} y \Big]_0^a = -V_0 \frac{2}{n\pi}\left((-1)^n - 1\right)$

$\Rightarrow a_n = 0 \quad n = \text{bik.}$

$\Rightarrow a_n = V_0 \frac{4}{n\pi} \quad n = \text{bak.}$

edo autofuntzio normalizatuak

$a_n = \left(V_0, \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} y\right)$

$\Rightarrow C_n e^{\frac{n\pi}{a}b} + D_n e^{-\frac{n\pi}{a}b} = V_0 \frac{a}{n\pi}\left((-1)^n - 1\right)$

$$\begin{cases} n = \text{bik} \rightarrow C_n e^{\frac{n\pi}{a}b} + D_n e^{-\frac{n\pi}{a}b} = 0 \rightarrow D_n = -C_n e^{\frac{2n\pi}{a}b} \quad (1) \\ n = \text{bak} \rightarrow C_n e^{\frac{n\pi}{a}b} + D_n e^{-\frac{n\pi}{a}b} = V_0 \frac{4}{n\pi} \quad (2) \end{cases}$$

• $\phi(-b,y)=\sum_{n=1}^{\infty}\left(C_n e^{-\frac{n\pi b}{a}}+D_n e^{\frac{n\pi b}{a}}\right)\sin\frac{n\pi}{a}y=V_0=\sum_{n=1}^{\infty}V_0\frac{a}{n\pi}((-1)^n-1)\sin\frac{n\pi}{a}y$

$$\Rightarrow \quad C_n e^{-\frac{n\pi b}{a}}+D_n e^{\frac{n\pi b}{a}} = V_0\frac{a}{n\pi}((-1)^n-1)$$

$$\begin{cases} n=\text{bik} \rightarrow C_n e^{-\frac{n\pi b}{a}}+D_n e^{\frac{n\pi b}{a}}=0 \rightarrow D_n=-C_n e^{-\frac{2n\pi b}{a}} \quad (3) \\ n=\text{bak} \rightarrow C_n e^{-\frac{n\pi}{a}b}+D_n e^{+\frac{n\pi b}{a}} = V_0\frac{4}{n\pi} \quad (4)\end{cases}$$

(2) eta (4) alderatuz $\Rightarrow$ $C_n e^{-\frac{n\pi}{a}b}+D_n e^{+\frac{n\pi b}{a}} = C_n e^{\frac{n\pi}{a}b}+D_n e^{-\frac{n\pi b}{a}} \rightarrow$

$$C_n\underbrace{\left(e^{\frac{n\pi}{a}b}-e^{-\frac{n\pi}{a}b}\right)}_{t}=D_n\underbrace{\left(e^{\frac{n\pi b}{a}}-e^{-\frac{n\pi b}{a}}\right)}_{t} \rightarrow t\neq 0 \ (n\neq 0) \Rightarrow C_n=D_n \quad (n=\text{bak.})$$

(1) eta (3) alderatuz $\Rightarrow$ $D_n=-C_n e^{-\frac{2n\pi b}{a}}=-C_n e^{\frac{2n\pi b}{a}} \rightarrow C_n=e^{\frac{4n\pi b}{a}}\cdot C_n \rightarrow$

$$C_n\left(1-e^{\frac{4n\pi b}{a}}\right)=0\ ,\quad n\neq 0 \Leftrightarrow C_n=D_n=0 \quad (n=\text{bik.}) \qquad *^2$$

$$\tilde{C}_m=2C_m$$

$$\phi(x,y)=\sum_{m=0}^{\infty}C_m\left(e^{(2m+1)\frac{\pi}{a}x}+e^{(2m+1)\frac{\pi}{a}x}\right)\sin(2m+1)\frac{\pi}{a}y=\sum_{m=0}^{\infty}\tilde{C}_m\cosh(2m+1)\frac{\pi}{a}x\ \sin(2m+1)\frac{\pi}{a}y$$

$\Rightarrow$ x-rekiko simetria: $\phi(x,y)=\phi(-x,y)$

$$\tilde{C}_n=2C_n$$

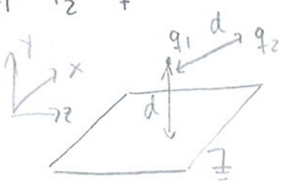
$$*^2 \quad C_n=D_n \ (n=\text{bak}) \Rightarrow \quad C_n\left(e^{\frac{n\pi b}{a}}+e^{-\frac{n\pi}{a}b}\right)=\tilde{C}_n\cosh\frac{n\pi b}{a}=V_0\frac{4}{n\pi} \Rightarrow$$

$$\tilde{C}_n=\frac{4\frac{V_0}{n\pi}}{\cosh\frac{n\pi b}{a}} \longrightarrow \tilde{C}_m=\frac{V_0\frac{4}{(2m+1)\pi}}{\cosh\frac{(2m+1)\pi b}{a}} \qquad m\in\mathbb{N} \Rightarrow$$

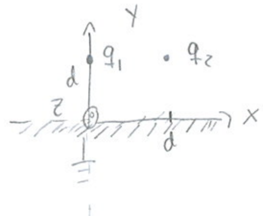
$$\boxed{\phi(x,y)=\sum_{m=0}^{\infty}\frac{V_0\,4\,\cosh\left[(2m+1)\frac{\pi}{a}x\right]}{(2m+1)\pi\cosh\left[(2m+1)\frac{\pi b}{a}\right]}\sin\left((2m+1)\frac{\pi}{a}y\right)}$$

2) Kalkula ezazu $q_2$ kargaren gainean sistemak (bi karga puntualak + lurrera lotuta dagoen planoa) eragindako indarra:

$q_1 = q_2 = q$

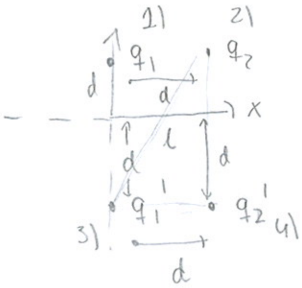



$\Rightarrow$




Karga induktarien metodoa erabiliz problema baliokidea ebatziko dugu:

Problema baliokidea (karga induktariak)




$q_1' = -q_1 = -q$, $q_2' = -q_2 = -q$

horrela, mugalde baldintza $\phi(x,0,z) = 0$ betetzen da.

- 1. karga $\Rightarrow$ $\phi_1 = \dfrac{q_1}{4\pi\varepsilon_0 r_1}$ ; 2. karga $\Rightarrow$ $\phi_2 = \dfrac{q_2}{4\pi\varepsilon_0 r_2}$ ; 3. karga $\Rightarrow$ $\phi_3 = \dfrac{q_1'}{4\pi\varepsilon_0 r_3} = \dfrac{-q_1}{4\pi\varepsilon_0 r_3}$

$r_1 = \sqrt{x^2 + (y-d)^2 + z^2}$ $\quad r_2 = \sqrt{(x-d)^2 + (y-d)^2 + z^2}$ $\quad r_3 = \sqrt{x^2 + (y+d)^2 + z^2}$

4. karga $\Rightarrow$ $\phi_4 = \dfrac{q_2'}{4\pi\varepsilon_0 r_4} = \dfrac{-q_2}{4\pi\varepsilon_0 r_4}$ $\Rightarrow$ $\phi(x,y,z) = \sum_{i=1}^{4} \phi_i(x,y,z)$

$r_4 = \sqrt{(x-d)^2 + (y+d)^2 + z^2}$

$\left( l = \sqrt{4d^2 + d^2} = \sqrt{5}\,d \right)$

$\phi(x,0,z) = 0$ betetzen da ✓

- 1, 3 eta 4 kargek 2. kargaren kontra eragindako indarra: $\vec F = \vec F_1 + \vec F_3 + \vec F_4 =$

$$\frac{q_2 q_1}{4\pi\varepsilon_0 d^2}\hat\imath + \frac{q_1' q_2}{4\pi\varepsilon_0 5d^2}\left(\frac{\hat\imath}{\sqrt5} + \frac{2\hat\jmath}{\sqrt5}\right) + \frac{q_2 q_2'}{4\pi\varepsilon_0 4d^2}\hat\jmath =$$

$q_2 = q_1 = q$, $q_1' = -q = q_2'$

$$\frac{q^2}{4\pi\varepsilon_0 d^2}\left(\hat\imath - \frac{\hat\imath}{5\sqrt5} - \frac{2\hat\jmath}{5\sqrt5} - \frac{\hat\jmath}{4}\right) = \frac{q^2}{4\pi\varepsilon_0 d^2}\left(\frac{(5\sqrt5-1)}{5\sqrt5}\hat\imath - \frac{(8+5\sqrt5)}{20\sqrt5}\hat\jmath\right) = \frac{q^2}{4\pi\varepsilon_0 d^2}(0'91\hat\imath - 0'43\hat\jmath)$$

3) Ingurune homogeneo, lineal, isotropo eta ez-magnetiko batean zehar hedatzen ari den uhin lau baten E eremu elektrikoaren adierazpena ondokoa dugu:

$$\vec E = 10 e^{-4z}\cos(30z - 10^9 t)\,\hat\imath \quad (V/m)$$

Lor bedi:

a) Uhin honen maiztasun lineala (frekuentzia), fase-abiadura eta uhin-luzera:

Uhin laua $\Rightarrow \vec{E} = \vec{E}_0 e^{-\beta z}\cos(\alpha z - \omega t + \theta_0)$, beraz gure adierazpenekin alderatuz,

$$\vec{E} = 10 e^{-4z}\cos(30z - 10^9 t)\,\hat{i} \quad (V/m)$$

↳ $\sigma \neq 0$ (material "eroale" batean)

Beraz:

$\omega = 10^9$ rad/s $\rightarrow \nu = \frac{\omega}{2\pi} = \frac{10^9}{2\pi}$ Hz $= 1'59\cdot 10^8$ Hz $= 159$ MHz

$\alpha = 30\ m^{-1} \Rightarrow v = \frac{\omega}{\alpha} = \frac{10^9\ s^{-1}}{30\ m^{-1}} = \frac{10^9}{30}$ m/s $= 3'33\cdot 10^7$ m/s

$\beta = 4\ m^{-1}$

$\theta_0 = 0 \Rightarrow \lambda = \frac{2\pi}{\alpha} = \frac{2\pi}{30}\ m = \frac{\pi}{15}\ m = 0'209\ m$

b) Uhin honen hedapen-zenbakia eta hedapen bektorea:

$K = \alpha + i\beta = |K| e^{i\Omega}$ $\quad |K| = \sqrt{\alpha^2+\beta^2} = 30'26\ m^{-1}$ $\quad \Omega = \arctg \frac{\beta}{\alpha} = 7'59° = 0'1325$ rad

$\vec{K} = |K| e^{i\Omega}\hat{k} = 30'26\, e^{0'133 i}\,\hat{k}$ $(m^{-1})$

↳ z-ren norantzko positiboan hedatu.

c) Uhin elektromagnetiko honi dagokion $\vec{B}$ indukzio magnetikoaren adierazpen bektoriala eta bien arteko desfasea:

E konplexua hartu.

$$\vec{B} = \frac{\vec{K}}{\omega}\times\vec{E} = \frac{|K|}{\omega} e^{i\Omega}\,\hat{k}\times\vec{E} = -\frac{|K|}{\omega} e^{i\Omega} E\,\hat{j} = +\frac{|K|}{\omega} e^{i\Omega}\, 10 e^{-4z} e^{i(30z-10^9 t)}\,\hat{j} =$$

$$3'026\cdot 10^{-7} e^{-4z} e^{i(30z-10^9 t+\Omega)}\,\hat{j} \Rightarrow \text{erreala} \Rightarrow \vec{B} = 3'026\cdot 10^{-7} e^{-4z}\cos(30z-10^9 t+\Omega)\,\hat{j} = \quad (T)$$

$$3'026\cdot 10^{-7}\cdot e^{-4z}\cos(30z-10^9 t+0'1325\ rad)\,\hat{j} \quad (T)$$

d) Uhin elektromagnetiko honi dagokion S Poynting bektorearen adierazpen bektoriala.

$$\vec{H} = \frac{\vec{B}}{\mu_0} = 0'24 e^{-4z}\cos(30z-10^9 t+0'1325)\,\hat{j} \quad (A/m)$$

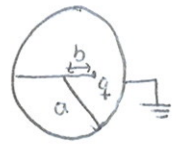
ingurune ez-magn.

$$\vec{S} = \vec{E}\times\vec{H} = 2'4 e^{-8z}\cos(30z-10^9 t+0'1325)\cos(30z-10^9 t+0'1325)\,\hat{k} \quad (W/m^2)$$

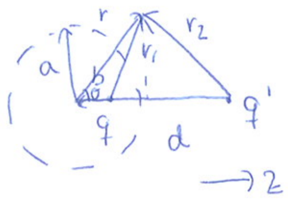
↳ Uhinengo bidebektu konplexuak eta gero erreala hartu?

2012-2013, 1. Partziala:

1.) Hutsik dagoen $a$ erradioko esfera eroalearen barruan $q$ balioko karga puntuala sartzen da, zentrotik $b$ distantziara. Esfera, lurrarekin lotuta dago. Kalkula itzazu potentziala eta eremua esferaren zentroan.



Problema baliokidea karga indukzioaren metodoaren bidez ⟹ eroalearen ordez karga indukzioa hartzen



Esfera eroalearen zentruan zentratutako koordenatu polarrak, simetriagatik $\varphi$ angeluaren independentea izango delako potentziala. $(r,\theta)$

$\vec{r}_1 = r\sin\theta\,\vec{j} + (r\cos\theta - b)\vec{i} \rightarrow r_1 = \sqrt{r^2\sin^2\theta + (r\cos\theta-b)^2} = \sqrt{r^2+b^2-2rb\cos\theta}$

$\vec{r}_2 = r\sin\theta\,\hat{j} + (r\cos\theta - d)\hat{i} \rightarrow r_2 = \sqrt{r^2\sin^2\theta + (r\cos\theta-d)^2} = \sqrt{r^2+d^2-2rd\cos\theta}$

Karga indukzioa zentrotik $d$ distantziara kokatu dugu eta haren $q'$ balioa eta posizioa kalkulatzeko mugalde baldintza aplikatuko dugu: $\phi(a,\theta)=0$

$$\underset{(r<a)}{\phi(r,\theta)} = \frac{q}{4\pi\varepsilon_0 r_1} + \frac{q'}{4\pi\varepsilon_0 r_2} \rightarrow \phi(a,\theta)=0 \leftrightarrow \phi(a,0)=\phi(a,\pi)=0$$

$$* \ \phi(a,0) = \frac{q}{4\pi\varepsilon_0\sqrt{a^2+b^2-2ab}} + \frac{q'}{4\pi\varepsilon_0\sqrt{a^2+d^2-2ad}} = \frac{q}{4\pi\varepsilon_0(a-b)} + \frac{q'}{4\pi\varepsilon_0(d-a)} = 0$$

$$q(d-a)+q'(a-b)=0 \rightarrow \cancel{q' = -q\frac{(a-d)}{(a-b)}} \quad (1)$$

$$* \ \phi(a,\pi) = \frac{q}{4\pi\varepsilon_0\sqrt{a^2+b^2+2ab}} + \frac{q'}{4\pi\varepsilon_0\sqrt{a^2+d^2+2ad}} = \frac{q}{4\pi\varepsilon_0(a+b)} + \frac{q'}{4\pi\varepsilon_0(a+d)} = 0$$

$$q(a+d)+q'(a+b)=0 \quad (2)$$

$$(1)+(2) \Rightarrow q(d-\cancel{a}+\cancel{a}+d) + q'(a-\cancel{b}+a+\cancel{b}) = q\cdot 2d + q'\cdot 2a = 0 \rightarrow q' = -q\frac{d}{a}$$

$(1)-(2) \Rightarrow \quad q(\cancel{\alpha}-a-a-\cancel{\alpha})+q'(\cancel{\alpha}-b-\cancel{\alpha}-b) = -2aq-2bq'=0 \rightarrow -aq = bq' \rightarrow$

$$q' = -\frac{a}{b}q = -\frac{d}{a}q \leftrightarrow \frac{a}{b} = \frac{d}{a} \rightarrow d = \frac{a^2}{b}$$

Beraz $q' = -\frac{a}{b}q$ eta $d = a^2/b$

Beraz, puntuala esferaren barnean $(r \leq a)$: $\phi(r,\theta) = \dfrac{q}{4\pi\varepsilon_0\sqrt{r^2+b^2-2rb\cos\theta}} + \dfrac{-\frac{a}{b}q}{4\pi\varepsilon_0\sqrt{r^2+\frac{a^4}{b^2}-\frac{2a^2}{b}r\cos\theta}}$

$$-\vec{\nabla}\phi = \vec{E} = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} = +\left[\frac{q}{4\pi\varepsilon_0}\frac{(r-b\cos\theta)}{(r^2+b^2-2rb\cos\theta)^{3/2}} - \frac{a/b\, q}{4\pi\varepsilon_0}\frac{(r-\frac{a^2}{b}\cos\theta)}{(r^2+\frac{a^4}{b^2}-\frac{2a^2}{b}r\cos\theta)^{3/2}}\right]\hat{r} +$$

$$\frac{1}{\cancel{r}}\left[\frac{q\,(\cancel{r}b\sin\theta)}{4\pi\varepsilon_0(r^2+b^2-2rb\cos\theta)^{3/2}} - \frac{a}{b}\frac{q}{4\pi\varepsilon_0}\frac{(\frac{a^2}{b}\cancel{r}\sin\theta)}{(r^2+\frac{a^4}{b^2}-\frac{2a^2}{b}r\cos\theta)^{3/2}}\right]\hat{\theta}$$

$$\vec{E}(0,\theta) = +\left[\frac{q}{4\pi\varepsilon_0}\frac{(-b\cos\theta)}{b^3} - \frac{a/b\,q}{4\pi\varepsilon_0}\frac{(-\frac{a^2}{b}\cos\theta)}{\frac{a^6}{b^3}}\right]\hat{r} + \left[\frac{q\,b\sin\theta}{4\pi\varepsilon_0 b^3} - \frac{a/b\,q}{4\pi\varepsilon_0}\frac{(\frac{a^2}{b}\sin\theta)}{\frac{a^6}{b^3}}\right]\hat{\theta} =$$

$$-\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{b^2}-\frac{b}{a^3}\right)\cos\theta\,\hat{r} + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{b^2}-\frac{b}{a^3}\right)\sin\theta\,\hat{\theta} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{b^2}-\frac{b}{a^3}\right)\underbrace{(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta})}_{=\hat{k}} =$$

$$\frac{qb}{4\pi\varepsilon_0}\left(\frac{1}{a^3}-\frac{1}{b^3}\right)\hat{k}$$

$$\phi(0,\theta) = \frac{q}{4\pi\varepsilon_0 b} + \frac{-a/b\,q}{4\pi\varepsilon_0\frac{a^2}{b}} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right)$$

2.) Irrati-antena baten sortutako 10 MHz-tako seinalearen argiaren intentsitatea $2\ mW/cm^2$-koa da. Neurketa hau antenatik oso urrun dagoen puntu batean eginda dago, eta orrat oner dezakegu hara heldutako uhina laua dela.

a) Zeintzuk dira eremu elektriko eta magnetikoaren anplitudeak?

$$\begin{cases} I = \langle S \rangle = 2\, mW/cm^2 = \dfrac{2\, \cancel{mW}}{\cancel{cm^2}} \cdot \dfrac{1\, W}{1000\, \cancel{mW}} \cdot \dfrac{1\, cm^2}{10^{-4}\, m^2} = 20 \cdot \dfrac{W}{m^2} \\ \nu = 10^{7}\, Hz \end{cases}$$

$$\vec{S} = \vec{E} \times \frac{\vec{B}}{\mu} = \vec{E} \times \vec{H} \Rightarrow \quad S = E\frac{B}{\mu} = \frac{E^2}{c\mu}$$

uhin laua eta suposatuz hutsean hedatzen dela $\rightarrow$ $\vec{B} = \dfrac{\vec{k}}{\omega} \times \vec{E} \rightarrow B = \dfrac{E}{c}$ $\;/\; (\vec{E} \perp \vec{B})$

$$\langle S \rangle = \frac{1}{2}\frac{E_0^2}{c\mu} = I \rightarrow E_0 = \sqrt{2Ic\mu} = 122'8\ V/m \qquad B_0 = \frac{E_0}{c} = 4'09 \cdot 10^{-7}\ T$$

$\rightarrow$ uhin laua $\left( \vec{E} = \vec{E}_0 \cos(kz - \omega t + \theta)\,,\ \vec{B} = \vec{B}_0 \cos(kz - \omega t + \theta) \right)$

$\Rightarrow$ Pertsona baten sarlu $\Rightarrow$ $\sigma = 4\ (\Omega m)^{-1}$, $\mu = \mu_0$, $\varepsilon_r = 80$ $(\varepsilon = \varepsilon_0 \varepsilon_r)$

b) Seinale horretarako eroale "onak" garela onar daiteke?

$$Q = \frac{\omega\varepsilon}{\sigma} = \frac{2\pi\nu\, \varepsilon_0 \varepsilon_r}{\sigma} = 1'11 \cdot 10^{-3} << 1 \Rightarrow \text{eroale } \underline{\text{onak}}$$

$$\omega = 2\pi\nu = 2\pi \cdot 10^{7}\ rad/s$$

c) Zenbat hedatuko da? Hedatu egingo da infinituraino baina eremuen amplitudea txikitzen joango da.

Sartzeakosuna: $\quad \delta = \dfrac{1}{\beta} \underset{Q<<1}{=} \dfrac{1}{\sqrt{\frac{1}{2}\mu_0 \sigma \omega}\,(1 - \frac{1}{2}Q)} = 8\ cm$

d) Zer balioak izango dira E eta B eremuek sarkorpen horretan?

$$z = \delta \rightarrow E = \frac{E_0}{e} = 44\ V/m \quad ; \quad B = \frac{B_0}{e} = 1'44 \cdot 10^{-7}\ T$$

3.) Frogatu a) hutseango Maxwell-en ekuazioak eta b) uhin elektromagnetiko lau baten energia dentsitatea eta Poynting bektorea, ondoko transformazioarekiko aldaezinak direla.

$$\vec{E}' = \vec{E}\cos\theta + c\vec{B}\sin\theta \quad ; \quad \vec{B}' = -(\vec{E}/c)\sin\theta + \vec{B}\cos\theta$$

a) Hutsengo Maxwellen ekuazioak:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{E} = 0 \;,\quad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$\hookrightarrow \rho = 0$ $\quad\hookrightarrow \vec{J} = 0$

- $\vec{\nabla}\times\vec{E}' = \vec{\nabla}\times(\vec{E}\cos\theta + c\vec{B}\sin\theta) = \vec{\nabla}\times\vec{E}\,(\cos\theta) + c\sin\theta\,\vec{\nabla}\times\vec{B} = -\frac{\partial\vec{B}}{\partial t}\cos\theta +$

$$c\sin\theta\,\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = -\frac{\partial\vec{B}}{\partial t}\cos\theta + \frac{\sin\theta}{c}\frac{\partial\vec{E}}{\partial t} = -\frac{\partial\vec{B}'}{\partial t}$$

$\hookrightarrow \vec{B}' = -(\vec{E}/c)\sin\theta + \vec{B}\cos\theta$

- $\vec{\nabla}\cdot\vec{E}' = \vec{\nabla}\cdot(\vec{E}\cos\theta + c\vec{B}\sin\theta) = \cos\theta\,\cancel{\vec{\nabla}\cdot\vec{E}}^{\,0} + c\sin\theta\,\cancel{\vec{\nabla}\cdot\vec{B}}^{\,0} = 0$

- $\vec{\nabla}\times\vec{B}' = \vec{\nabla}\times(\vec{B}\cos\theta - \frac{\vec{E}}{c}\sin\theta) = \cos\theta\,\vec{\nabla}\times\vec{B} - \frac{\sin\theta}{c}\vec{\nabla}\times\vec{E} = \cos\theta\,\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} +$

$$\frac{\sin\theta}{c}\frac{\partial\vec{B}}{\partial t} = \mu_0\varepsilon_0\left(\cos\theta\frac{\partial\vec{E}}{\partial t} + \frac{\sin\theta}{c\mu_0\varepsilon_0}\frac{\partial\vec{B}}{\partial t}\right) = \mu_0\varepsilon_0\left(\cos\theta\frac{\partial\vec{E}}{\partial t} + \sin\theta\,c\frac{\partial\vec{B}}{\partial t}\right) =$$

$$\mu_0\varepsilon_0\frac{\partial\vec{E}'}{\partial t}$$

- $\vec{\nabla}\cdot\vec{B}' = \vec{\nabla}\cdot(\vec{B}\cos\theta - \frac{\vec{E}}{c}\sin\theta) = \cos\theta\,\cancel{\vec{\nabla}\cdot\vec{B}}^{\,0} - \frac{\sin\theta}{c}\cancel{\vec{\nabla}\cdot\vec{E}}^{\,0} = 0$

b) Energia dentsitatea:

$$\frac{du}{dv} = \frac{1}{2}\varepsilon_0E^2 + \frac{1}{2\mu_0}B^2 \qquad (\text{uhin-laua})$$

$\vec{B}\perp\vec{E}$

- $E'^2 = E^2\cos^2\theta + c^2B^2\sin^2\theta + \cancel{2\vec{E}\cdot\vec{B}\cdot c\sin\theta\cos\theta}$

$\vec{E}\perp\vec{B}$

- $B'^2 = B^2\cos^2\theta + \frac{E^2}{c^2}\sin^2\theta + \cancel{2\vec{E}\cdot\frac{\vec{B}}{c}\sin\theta\cos\theta}$

$c^2 = \frac{1}{\varepsilon_0\mu_0}$

$$\frac{du'}{dv} = \frac{1}{2}\varepsilon_0E'^2 + \frac{1}{2}\mu_0B'^2 = \frac{E^2}{2}\cos^2\theta\,\varepsilon_0 + \frac{\varepsilon_0}{2}c^2B^2\sin^2\theta + \frac{1}{2\mu_0}B^2\cos^2\theta + \frac{1}{\mu_0 2c^2}E^2\sin^2\theta =$$

$$\frac{E^2}{2}\varepsilon_0\underbrace{(\cos^2\theta+\sin^2\theta)}_{\substack{\shortparallel\\ 1}} + \frac{1}{2}\frac{B^2}{\mu_0}\underbrace{(\sin^2\theta+\cos^2\theta)}_{\substack{\shortparallel\\ 1}} = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 = \frac{du}{dv}$$

$$\vec S = \vec E \times \vec H = \vec E \times \frac{\vec B}{\mu}$$

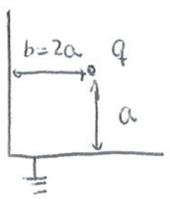
$$\vec S' = \vec E' \times \vec H' = \vec E' \times \frac{\vec B'}{\mu} = (\vec E\cos\theta + c\vec B\sin\theta)\times \frac{1}{\mu}\left(\vec B\cos\theta - \frac{\vec E}{c}\sin\theta\right) =$$

$$\vec E\times\vec B\left(\frac{\cos^2\theta}{\mu}\right) - \frac{\cos\theta\sin\theta}{c}\underbrace{\cancel{\vec E\times\vec E}}_{\vec E \parallel \vec E} + \frac{c}{\mu}\sin\theta\cos\theta \underbrace{\cancel{\vec B\times\vec B}}_{\vec B\parallel\vec B} - \frac{1}{\mu}\vec B\times\vec E\sin^2\theta =$$

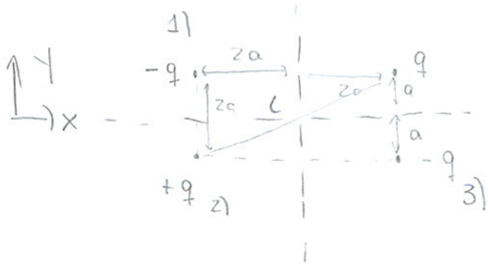
$$\frac{1}{\mu}\cos^2\theta\, \vec E\times\vec B + \frac{1}{\mu}\sin^2\theta\,\vec E\times\vec B = \frac{1}{\mu}\vec E\times\vec B = \vec S$$

## 2013 Uztaila:

1.1)




Problema baliokidea ⇒ Karga irudien metodoa




$$\begin{cases} l = \sqrt{4a^2+16a^2} = a\sqrt{20} = 2a\sqrt5 \\ \vec l = 2a\hat j + 4a\hat i \\ \hat l = \dfrac{2a\hat j + 4a\hat i}{2a\sqrt5} = \dfrac{\hat j}{\sqrt5}+\dfrac{2\hat i}{\sqrt5} \end{cases}$$

q-ren gainean planoak egindako indarra ⟺ Karga irudiek egindako indarra:

$$\vec F = \vec F_1 + \vec F_2 + \vec F_3 = \frac{q}{4\pi\varepsilon_0}\left( \frac{-q}{(4a)^2}\hat i - \frac{q}{(2a)^2}\hat j + \frac{q}{(a\sqrt{20})^2}\left(\frac{\hat j}{\sqrt5} + \frac{2\hat i}{\sqrt5}\right)\right) =$$

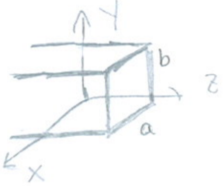
$$\frac{q^2}{16\pi\varepsilon_0 a^2}\left(\frac{-\hat i}{4} - \hat j + \frac{\hat j}{5\sqrt5} + \frac{2\hat i}{5\sqrt5}\right) = \frac{q^2}{16\pi\varepsilon_0 a^2}\left(\left(\frac{2}{5\sqrt5} - \frac{1}{4}\right)\hat i + \left(\frac{1}{5\sqrt5} - 1\right)\hat j\right) =$$

$$\frac{q^2}{16\pi\varepsilon_0 a^2}\left(-0'071\,\hat i - 0'91\,\hat j\right) = \frac{q^2}{4\pi\varepsilon_0 a^2}\left(-0'01778\,\hat i - 0'22764\,\hat j\right) \quad \text{(erakarlea)}$$

3)

Irrati uhinetarako ⇒ errepidetako tunelak eroale ia perfektuez inguratutako gidak.

Sekzio errektangularra: $a = 8\,m$, $b = 5\,m$




Ebakidura maiztasuna: $\omega_{mn} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

$$\nu_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2\pi}\pi\sqrt{\left|\frac{m}{a}\right|^2+\left|\frac{n}{b}\right|^2} = 1'5\cdot 10^8\sqrt{\left|\frac{m}{a}\right|^2+\left|\frac{n}{b}\right|^2}\ m/s$$

Indargetzena: $m=1, n=0 \rightarrow$ $\nu_{10} = \frac{1'5\cdot 10^8}{8}\ s^{-1} = 18'75\ MHz$

Heda dadin $\nu > \nu_{mn}$: $\searrow TE_{10}$

a) AM irrati-uhinak: $\nu = 1000\ kHz = 10^6\ Hz = 10^6\ s^{-1}$

$$\nu \geq \nu_{mn} \rightarrow 10^{12}\ s^{-2} \geq 2'25\cdot 10^{16}\ m^2 s^{-2}\left(\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}\right)^2 = 2'25\cdot 10^{16}\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right] \Rightarrow$$

$$4'44\cdot 10^{-5}\ m^{-2} \geq \frac{m^2}{a^2}+\frac{n^2}{b^2} = \frac{m^2}{64}+\frac{n^2}{25}\ m^{-2}$$ ⇒ ezinezkoa:

$m$ eta $n$ osoak izanda ere ez da hedatuko $(m=0, n=1;\ m=1, n=0)$

b) Taxiko banda: $\nu = 27\,MHz = 2'7\cdot 10^7\ Hz = 2'7\cdot 10^7\ s^{-1}$

$$\nu \geq \nu_{mn} \rightarrow (2'7\cdot 10^7)^2\ s^{-2} \geq 2'25\cdot 10^{16}\ m^2 s^{-2}\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right] \rightarrow$$

$$3'24\cdot 10^{-2}\ m^{-2} \geq \frac{m^2}{a^2}+\frac{n^2}{b^2} = \frac{m^2}{64}+\frac{n^2}{25}\ m^{-2}$$ ⇒ Bai $m=1$ eta $n=0$

egiten badugu ⇒ $TE_{10}$ modoa

$$k_g^2 = k_0^2 - \left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right] \underset{m=1,\ n=0;\ k_0=\left(\frac{2\pi\nu}{c}\right)}{=} k_0^2 - \left(\frac{\pi}{a}\right)^2 = \left(\frac{2\pi\nu}{c}\right)^2 - \frac{\pi^2}{a^2} \rightarrow k_g = \frac{2\pi}{\lambda_g} \rightarrow$$

$$\lambda_g = \frac{2\pi}{k_g} = 15'44\ m$$

c) FM irrati-uhinak: $\nu = 100\ MHz = 10^8\ Hz = 10^8\ s^{-1}$

$$\nu \geq \nu_{mn} \Rightarrow \quad \nu^2 \geq \nu_{mn}^2 \rightarrow 10^{16}\, s^{-2} \geq 2'25 \cdot 10^{16}\, m^{-2} s^{-1} \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right] \Rightarrow$$

$$0'444\, m^{-2} \geq \frac{m^2}{64} + \frac{n^2}{25}\, m^{-2} \quad ,$$

TE$_{10}$, TE$_{01}$, TE$_{11}$, TE$_{21}$, TE$_{12}$, TE$_{22}$, TE$_{32}$, TE$_{23}$, TE$_{31}$, TE$_{13}$, TE$_{02}$, TE$_{20}$, TE$_{30}$, TE$_{03}$, TE$_{40}$, TE$_{04}$, TE$_{41}$, TE$_{14}$, TE$_{42}$, TE$_{24}$, TE$_{43}$, TE$_{34}$, TE$_{44}$, TE$_{50}$, TE$_{05}$, TE$_{51}$, TE$_{15}$, TE$_{52}$, TE$_{25}$, TE$_{53}$, TE$_{35}$, TE$_{54}$, TE$_{45}$, TE$_{55}$, TE$_{60}$, TE$_{06}$, TE$_{61}$, TE$_{16}$, TE$_{62}$, TE$_{26}$, TE$_{63}$, TE$_{36}$, TE$_{64}$, TE$_{46}$, TE$_{65}$, TE$_{56}$, TE$_{66}$, TE$_{70}$, TE$_{07}$, TE$_{71}$, TE$_{17}$, TE$_{72}$, TE$_{27}$, TE$_{73}$, TE$_{37}$, TE$_{74}$, TE$_{47}$, TE$_{75}$, TE$_{57}$, TE$_{76}$, TE$_{67}$, TE$_{77}$, TE$_{80}$, TE$_{08}$, TE$_{81}$, TE$_{18}$, TE$_{82}$, TE$_{28}$, TE$_{83}$, TE$_{38}$, TE$_{84}$, TE$_{48}$, TE$_{85}$, TE$_{58}$, TE$_{86}$, TE$_{68}$, TE$_{87}$, TE$_{78}$, TE$_{90}$, TE$_{09}$, TE$_{91}$, TE$_{19}$, TE$_{92}$, TE$_{29}$, TE$_{93}$, TE$_{39}$, TE$_{94}$, TE$_{49}$, TE$_{95}$, TE$_{59}$, TE$_{96}$, TE$_{97}$...

$m=1$, $n=0$ TE$_{10}$ hedatzen bada:

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2} \rightarrow$$

$$K_g^2 = K_0^2 - K_c^2 = \left( \frac{2\pi\nu}{c} \right)^2 - \left( \frac{\pi}{a} \right)^2 \rightarrow$$

$$K_g = \frac{2\pi}{\lambda_g} \rightarrow \lambda_g = \frac{2\pi}{K_g} = 3'054\, m$$

Galderak:

erradiazio presioa

1.) 4a) $I = 1400\, W/m^2$, lurra erabat xurgatzailea $(R=0)$. $P_r$?

Suposatu erasoa normala dela. $P_r = \langle u_e \rangle = \frac{\langle S \rangle}{c} = \frac{I}{c} = 4'666\, N/m^2$

2.) 4b) Uhin e-m lauak: $\nu = 1\, GHz = 10^9\, Hz \rightarrow$ kupreztko gorputza perpendikularki jo.

$\sigma = 5'8 \cdot 10^7\, (\Omega m)^{-1}$, $\mu = \mu_0$. $\lambda$? $\varepsilon = \varepsilon_0$

perfektua

$\lambda = \frac{2\pi}{\alpha}$ , $Q = \frac{\omega\varepsilon}{\sigma} = \frac{2\pi\nu\varepsilon}{\sigma} = 9'587 \cdot 10^{-10} \ll 1 \Rightarrow$ eroale oso ona

$\alpha = \beta = \sqrt{\frac{\mu\sigma\omega}{2}} = 4'785 \cdot 10^5\, m^{-1}$, $\lambda = 1'31 \cdot 10^{-5}\, m \simeq 13\, \mu m$

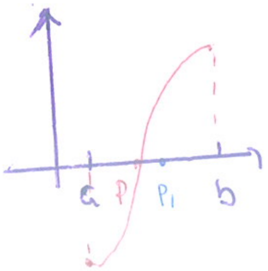
$\lambda_{hutsean} = \frac{c}{\nu} = 0'3 m$

## METODO KONPUTAZIONALAK:

16-11-29

1) Bolzano: $f(x)$ jarraia bada $[a,b]$ tartean, eta $f(a)\,f(b)<0$ bada,

$$\exists p \in [a,b] \;/\; f(p)=0$$




Gure input: $[a,b]$, funtzio baten erroa aurkitzeko tarte batean.

$p_1 = \frac{b+a}{2}$ erdiko puntua hartu eta ikusi ea $f(p_1)=0$. $f(p)>0$ bada eta $f(b)>0$, $[a,p_1]$ tartea hartu, edo alderantziz, $f(p_1)<0$ bada $[p_1,b]$ hartu. $\Rightarrow$ jarraitu prozesu berdina: $p_2 = \frac{a+p_1}{2}$ edo $p_2 = \frac{b+p_1}{2}$ ...

$$[a,b] \Rightarrow \left[a, \frac{a+b}{2}\right] \left[\frac{a+b}{2}, b\right]$$

$f(a)\cdot f\left(\frac{a+b}{2}\right) > 0$ (bada tarte honetan ez.) $\Rightarrow \left[\frac{a+b}{2}, b\right]$ tartean. $\Rightarrow$

$$a \to \frac{a+b}{2}$$

$$b \to b$$

else.

$$a \to a$$

$$b \to \frac{b+a}{2}$$

iterazio kopurua

ln-ak oso azkarrak.

$[a,b]$, $\Delta_0 = (b-a)$ $(b>a)$ $\Rightarrow$ tartea

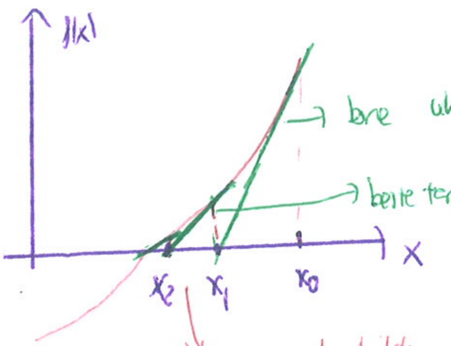
$\Delta_n = \frac{\Delta_0}{2^n} \to \varepsilon \simeq \Delta_0 \cdot 2^{-n}$, $\ln|\varepsilon| \simeq \ln|\Delta_0| - n\ln(2) \Rightarrow n \simeq \frac{\ln|\Delta_0| - \ln|\varepsilon|}{\ln 2} = \frac{\ln|\Delta_0/\varepsilon|}{\ln 2}$

errorea? zehaztasuna

Zehaztasuna, $\varepsilon$, $\simeq \Delta_n$-ren antzekoa denean $\Rightarrow$ gelditu

ERDIBIKETA METODOA

## 2.) NEWTONEN METODOA: (erabilgarrena denean, ez baita metodo arruntena da)




Ukitzailea ⇒ 1. ordenako Taylor-en garapena

$$f(x) \Rightarrow f(x_0) + (x - x_0)\, f'(x_0)$$

↓ gai izan behar gara funtzioaren deribatua kalkulatzen

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad n \geq 1 \Leftarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

( $f(x_0) + (x - x_0)\, f'(x_0) = 0$ denean)

Arazoa. $f(x)$, $f'(x)$, $x_0$ behar dugu : $f(x)$ jarraia izan behar da, $f'(x)$ existitu behar da, $f(x)$ ezagutu behar da, $x_0$ posizio bat...

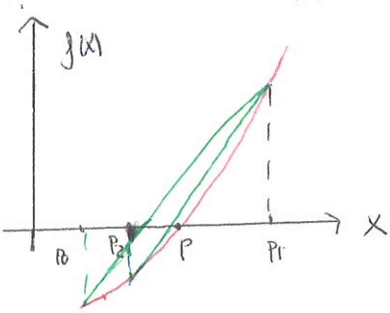
→ errotik ez oso urrun.

↓ batzuetan analitikoki ez da ezagutzen.

Arazoa $f'(x) = 0$ bada!

16-12-2

## 3.) EBAKITZAILEEN METODOA: (Secant method)




$p_0$, $p_1$ (bi puntu), $f(x)$

Newtonen: $p_{n+1} = p_n - \dfrac{f(p_n)}{f'(p_n)}$ ⇒ segida bat lortu, orduan limitea (errora aurkitzen $f(p_n) = 0$ bada)

Metodo honetan bidez $f'(p_n)$-ren hurbilketa egingo dugu.

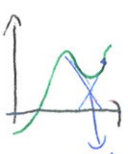
$$f'(p_n) \simeq \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}} = \frac{\Delta f}{\Delta p}$$

(Horregatik behar ditugu bi puntu)

Orduan ⇒ $p_n = p_{n-1} - \dfrac{f(p_{n-1})\,(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$ $n \geq 2$

Arazoak: Newton:

Funtzio guztietarako ezin da aplikatu



ez da inoiz konbergituko, funtzioak minimo bat duelako hor, ahurra da

Laburbilduz:

1) Erdibiketa $[a, b, f(x)]$
2.) Newton $[x_0, f(x), f'(x)]$
3.) Ebakitzailea $[x_0, x_1, f(x)]$

} Egin Fortran-en modulu bat honela gordeta izateko.

Hobeto funtzioak erabiltzea soluzio balioa ematen duelako: (FORTRAN-en)

1.) Interface bat erabili behar dugu aldagaia funtzio bat delako

```
function erdibiketa (a, b, f)
  real, intent (in) :: a, b
  interface
     function f(x)
        real, intent (in) :: x
        real :: f
     end function f
  end interface
```

16-12-16

## INTERPOLAZIOA eta ESTRAPOLAZIOA.

Interpolazioa ⇒ datu batzuk emanik neurketako tartean funtzioak izan ditzakeen balioen estimazioa

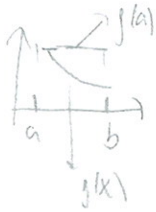
Estrapolazioa ⇒ Neurketako tartetik kanpora estimazio bat egitea

→ Interpolazio polinomikoa.

**0 ordenako interpolazioa:**

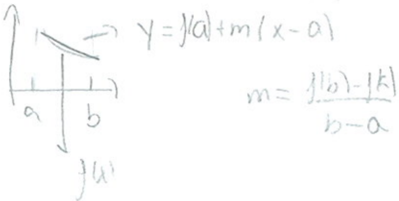
↳ puntu bat bakarrik.

f funtzioa, (a, b) tartean ⇒ f(a) hartu tarte osoan (lerro bat)




**1. ordenako interpolazioa:**

f(a) eta f(b) ⇒ zuzen bat erabili bi puntu horiekin




**2. ordenako interpolazioa:**

$f(a)$, $f(x_1)$ eta $f(b)$ ⇒ interpolazioa puntu horietatik pasatu behar da ⇒ interpolazio polinomiala ⇒ parabola bat erabili: $y = px^2+qx+r = f(x)$

↓ tarteko edozein puntu.

Datuak: $(a, f(a)), (x_1, f(x_1)), (b, f(b))$

⇒ $f(a) = pa^2+qa+r$, $f(x_1) = px_1^2+qx_1+r$, $f(b) = pb^2+qb+r$ ⇒ ebatzi eta p, q, r lortu

Soluzioa ⇒ $$f(x) = \frac{(x-x_1)(x-b)}{(a-x_1)(a-b)} f(a) + \frac{(x-a)(x-b)}{(x_1-a)(x_1-b)} f(x_1) + \frac{(x-a)(x-x_1)}{(b-a)(b-x_1)} f(b)$$

↳ eraldaketa batzuk egin

## Lagrange-ren polinomioak (interpolazioa)

Edozein dimentsiorekin emaitza lortzeko (ez ebazteko sistema lineala)

→ interpolatzeko funtzioa

2. ordenean ⇒ $\tilde{f} = p_1(x) f(a) + p_2(x) f(x_1) + p_3(x) f(b)$

$\tilde{f}(a) = f(a)$ denez $p_2(a)$ eta $p_3(a)$ 0 izan behar dira eta $p_1(a) = 1$

↳ nodoak.

Gauza bera b eta $x_1$-ekin ⇒ baldintza hauek betetzen dituen polinomio bakarra.

Erregela orokorra: Lagrange-ren polinomioa:

n. ordena

k. puntuarekiko

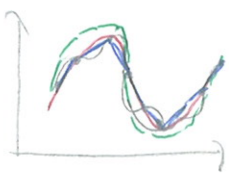
$$\tilde{f} = P_n(x) = \sum_{k=0}^{n} f(x_k)\, L_{n,k}(x) \quad ; \quad L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

ordena

n+1 puntu

Aplikazioa ⇒ zenbakizko integrazioa eta deribatuen zenbakizko kalkulua

* Maila handiko polinomioak oso oszilakorrak dira ⇒ Batzuetan hobeto zatika maila baxuagoko interpolazio batzuk egitea



- Benetakoa.
- 8. ordenekoa.
- 4. ordenekoa ⇒ 2 interpolazio
- 1. ordenekoa. ⇒ 8 interpolazio.

8 puntu ditugu

$x \in [x_j, x_{j+1}]$

$S_j(x) = a_j (x - x_j)^3 + b_j (x - x_j)^2 + c_j (x - x_j) + d_j$

Funtzioa tarte ezberdinetan banatu eta bi puntuen arteko interpolazioa 3. mailako polinomio baten bidez (4 koef)

$$\begin{cases} S_j(x_j) = f(x_j) & j = 0, 1, \dots, n \\ S_{j+1}(x_{j+1}) = S_j(x_{j+1}) & j = 0, 1, \dots, n-2 \\ S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}) & j = 0, 1, \dots, n-2 \\ S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}) & j = 0, 1, \dots, n-2 \end{cases}$$

→ n+1 puntu

→ jarraia izateko, puntu beretik pasatzeko.

1. eta 2. deribatuak jarraiak!

\+ Bi baldintza sistema osatzeko

$$\begin{cases} S''(x_0) = S''(x_n) = 0 \\ S'(x_0) = f'(x_0), \quad S'(x_n) = f'(x_n) \end{cases}$$

"Spline" naturala edo "askea"

"Spline" mugatua

e(n) ⇒ gif-ak

$L_{n,i}(x) = \prod_{\substack{k=0 \\ i\neq k}}^{n} \frac{(x-x_k)}{(x_i-x_k)}$

# METODO KONPUTAZIONALAK 2. KUATRIA:

17-02-03

polinomioaren hurbilduz: $f(x) \simeq \underbrace{\sum_i f(x_i) L_{n,i}(x)}_{P_n(x)}$

## Koadratura formula:

(Newton Cotes itxoak)

$$\int_a^b f(x)\,dx = \sum_i w_i f(x_i)$$


* Adibidez $n=2 \Rightarrow f(x) \simeq f_0 L_{2,0}(x) + f_1 L_{2,1}(x) + f_2 L_{2,2}(x)$

↳ $n+1 = 3$ puntu $x_0, x_1, x_2$ $\quad x \in [x_0, x_2]$

$$\int_{x_0}^{x_2} f(x)\,dx \simeq f_0 w_0 + f_1 w_1 + f_2 w_2$$

$|x_1 - x_0| = |x_2 - x_1| = h$ (pausua)

$$w_0 = \int_{x_0}^{x_2} dx\, L_{2,0}(x) = \int_{x_0}^{x_2} dx\, \frac{(x-h)(x-2h)}{(0-h)(0-2h)} = \frac{h}{3} \; ; \quad w_1 = \int_{x_0}^{x_2} dx\, L_{2,1}(x) = \int_{x_0}^{x_2} dx\, \frac{(x-0)(x-2h)}{(h-0)(h-2h)} = \frac{4h}{3}$$

$$w_2 = \int_{x_0}^{x_2} dx\, L_{2,2}(x) = \int_{x_0}^{x_2} dx\, \frac{(x-0)(x-h)}{(2h-0)(2h-h)} = \frac{h}{3} \Rightarrow \int_{x_0}^{x_2} f(x)\,dx \simeq \frac{h}{3}(f_0 + 4f_1 + f_2) + O(h^5)$$

Simpson-en erregela

* $n=1 \Rightarrow \int_a^b f(x)\,dx \simeq \frac{h}{2}[f(x_0) + f(x_1)] + O(h^3)$ Trapezioen erregela

* $n=3 \Rightarrow \int_a^b f(x)\,dx \simeq \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] + O(h^5)$ Simpson-en 3/8 erregela

* $n=4 \Rightarrow \int_a^b f(x)\,dx \simeq \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] + O(h^7)$

↳ Simetrikoki koheziala koefizienteak

Boole-ren erregela

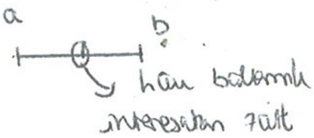
k koefizienteen batura beti da $n \cdot h \Rightarrow f_i = 1 \Rightarrow \int_{x_0}^{x_n} 1 = h \cdot n \cdot 1 = nh$

## Newton-Cotes formula irekiak:

pausua ↷ $h = (b-a)/(n+2)$ $\quad (x_{-1} = a,\ x_0 = a+h, \ldots, x_{n+1} = b)$

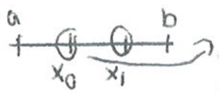
↳ a eta b ez zaizkit interesatzen $x_{-1}$ eta $x_{n+1}$

- $n=0$




$$\int_a^b f(x)\,dx \simeq 2h\, f(x_0) + O(h^3)$$

- $n=1$




$$\int_a^b f(x)\,dx \simeq \frac{3h}{2}[f(x_0) + f(x_1)] + O(h^3)$$

Koefizienteen batura $(n+2)h$

- $n=2$ $\quad \int_a^b f(x)\,dx \simeq \frac{4h}{3}[2f(x_0) - f(x_1) + 2f(x_2)] + O(h^5)$

- $n=3$ $\quad \int_a^b f(x)\,dx \simeq \frac{5h}{24}[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)] + O(h^5)$

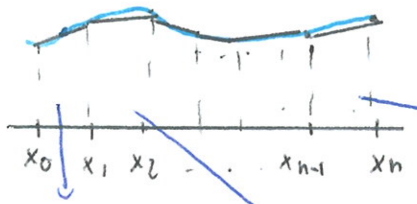
## Trapezioaren erregela konposatua:

- $[a,b]$ tartea $m$ tarte txikitan banatu $\Rightarrow$ $h=(b-a)/m$
- Trapezioaren erregela aplikatu tarte txiki bakoitzean $\Rightarrow$ $[a+jh,\ a+(j+1)h]$ $\quad j=0,\dots,m-1$

$$I_j=\frac{h}{2}[f(a+jh)+f(a+(j+1)h)]$$

m ordenako polinomioa erabili beharrean tarte txikitan banatu eta orden txikiko polinomioekin hurbildu
$\Downarrow$
trapezioaren erregela edo Simpson-en erregela aplikatu

- $[a,b]$ tarte osoaren integrala: $I=\sum_{j=1}^{m} I_j = \frac{h}{2}\{f(a)+2\sum_{j=1}^{m-1} f(a+jh)+f(b)\}$




$\frac{h}{2}(f(x_{n-1})+f(x_n))$

$\frac{h}{2}(f(x_0)+f(x_1))$ $\qquad$ $\frac{h}{2}(f(x_1)+f(x_2))$

## Simpson-en erregela konposatua: ($h^4$ errorea)

- $[a,b]$ tartea $m$ zatitan banatu ($m$ bikoitia) $h=(b-a)/m$ zabalerako tartetan.
  ↳ puntu kopuru bakoitia
- Tarte bakoitzean Simpson-en erregela aplikatu $\Rightarrow$ $[a+jh,\ a+(j+2)h]$ erdiko puntuak: $a+(j+1)h$ $\quad (j=0,\dots,m-2)$

Simpson-en erregela: $I_j=\frac{h}{3}\{f(a+jh)+4f[a+(j+1)h]+f(a+(j+2)h)\}$

- $[a,b]$ tarte osoaren integrala: $I=\sum_{j=0}^{m/2-1} I_{2j} = \frac{h}{3}\{f(a)+2\sum_{j=1}^{\frac{m}{2}-1} f(a+2jh)+4\sum_{j=1}^{m/2} f(a+(2j-1)h)+f(b)\}$

↳ bikoitiak $\qquad$ bakoitiak

17-02-07 **INDUKZIOA**

Suposatu horrela dela

$$M-N(h)=K_1h+K_2h^2+K_3h^3+\dots \quad (\text{errorea})$$

N(h) funtzioak integralaren hurbilpena ematen du

$$M=N(h)+K_1h+K_2h^2+K_3h^3+\dots$$

$h\to h/2$ eginez berdin: $\quad M=N\left(\frac{h}{2}\right)+K_1\frac{h}{2}+K_2\left(\frac{h}{2}\right)^2+K_3\left(\frac{h}{2}\right)^3+\dots$

(pausua)

Biak konbinatuz:

$$M = \left[ N\left(\frac{h}{2}\right) + \left(N\left(\frac{h}{2}\right) - N(h)\right) \right] + K_2 \left(\frac{h^2}{2} - h^2\right) + K_3 \left(\frac{h^3}{4} - h^3\right) + \dots$$

Integralaren zehaztasunak "ordena magnitude bat" irabazi du

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[ N_1\left(\frac{h}{2}\right) - N_1(h) \right] \quad ; \quad M = N_2(h) - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 \dots \quad (1)$$

Trapezioaren metodoa baino hobea

$h \to h/2$ eginez berriz: $M = N_2\left(\frac{h}{2}\right) - \frac{K_2}{8} h^2 - \frac{3K_3}{32} h^3 \dots \quad (2)$

Bi hauek berriz konbinatuz: $4(2) - (1) \Rightarrow 3M = 4N_2\left(\frac{h}{2}\right) - N_2(h) +$

$$3\frac{K_3}{4}\left(-\frac{h^3}{2} + h^3\right) + \dots \Rightarrow M = \left[ N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{3} + \frac{K_3}{8} h^3 \right]$$

Ordena magnitudea igo berriz. $\Rightarrow$ hau etengabe aplika daiteke:

$$N_j\left(\frac{h}{2}\right) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{2^{j-1} - 1}$$

ROMBERG:

trapezioaren metodoa (erregularra da bikoitiak direla)

$$\int_a^b f(x)\,dx - R_{K,1} = \sum_{i=1}^{\infty} K_i h_K^{2i} = K_1 h_K^2 + \sum_{i=2}^{\infty} K_i h_K^{2i}$$

Berretzaile bikoitiak baditu

integralaren hurbilketa

errorea

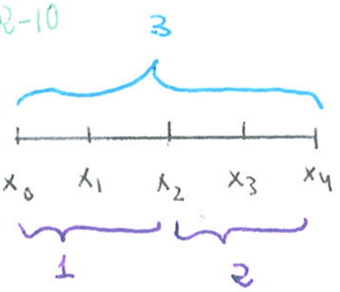
$K-1 \to K$ $\left(\frac{h}{2}\right)$ egin

$$R_{K,j} = R_{K,j-1} + \frac{R_{K,j-1} - R_{K-1,j-1}}{4^{j-1} - 1} \quad \text{(Romberg)}$$

$\left( \begin{array}{l} R_{1,1} \to \text{trapezio} \\ R_{2,2} \to \text{Simpson} \\ R_{3,3} \to \text{Boole} \end{array} \right)$

$K \Rightarrow 2^{K-1}+1$ puntu ($R_{K,1} \to$ trapezio konposatua $2^{K-1}+1$ puntuekin)

17-02-10




$I(h) = I_0 + I_2 h^4 \quad ; \quad I(2h) = I_0 + 16 I_2 h^4$

$16I(h) - I(2h) = 15 I_0 \Rightarrow I_0 = \frac{16I(h) - I(2h)}{15}$

Simpson konposatua $n=4$

$I(h) = I_1 + I_2 = \frac{h}{3}\{f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)\}$

$h = x_1 - x_0 = x_i - x_{i-1}$ $\quad i = 1, 4$

$I(2h) = \frac{2h}{3}\{f(x_0) + 4f(x_2) + f(x_4)\}$ Simpson konposatua $m=2$

$$I_0 = \frac{h}{45}\{14 f(x_0) + 64 f(x_1) + 24 f(x_2) + 64 f(x_3) + 14 f(x_4)\}$$ Bode-ren erregela

17-02-14

# Gauss koadratura

Gauss koadraturaren oinarrizko ideia: $\int_a^b f(x)\,dx \simeq \sum_{k=1}^{n} w_k f(x_k)$

Aukeratu $w_k$ eta $x_k$ (guztira $2n$ parametro) hurrengo baldintzarekin: (zehaztasun gradurik altuena → puntu kopurua)

Lortutako integrala zehatza izatea edozein $2n-1$ ordeneko polinomioentzat. (↳ $2n$ zehaztasun gradua)

Demostratu daiteke aurreko baldintzak praktikan $f$-ren menpekoa ez dela ($f(x)$ eta tartearen arabera) "$x_k$ puntuak" polinomio familia jakin batzuetako zeroak baldin badira.

Ad. $n=2 \Rightarrow \int_a^b f(x)\,dx \simeq w_1 f(x_1) + w_2 f(x_2)$

3 ordeneko polinomioa $\Rightarrow P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

Polinomio horietarako zehatza izan behar da integrala:

$$\int_a^b f(x)\,dx \simeq w_1 p(x_1) + w_2 p(x_2) = (b-a)c_0 + \frac{c_1}{2}(b^2-a^2) + \frac{c_2}{3}(b^3-a^3) + \frac{c_3}{4}(b^4-a^4)$$ (zehatza izateko)

* Trapezioaren erregelarekin integrala baliomile zen zehatza zuzenekin → metodo horiekin polinomioekin → zehaztasun gradu maximoa ($x_k$ zehaztu gabe; aske)

- Oinarri bat hartu → $\{1, x, x^2, x^3\}$ horri edozein konbinazio 3. ordeneko polinomioa da. $1 \to \int_{-1}^{1} 1\,dx = c_1 \cdot 1 + c_2 \cdot 1 = x\Big]_{-1}^{1} = 2 = c_1 + c_2$

$x \to \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big]_{-1}^{1} = 0 = c_1 x_1 + c_2 x_2$ * $a=-1$, $b=1$ tartea hartuz

$x^2 \to \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3} = c_1 x_1^2 + c_2 x_2^2$

$x^3 \to \int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big]_{-1}^{1} = 0 = c_1 x_1^3 + c_2 x_2^3$

3 ordenako edozein polinomioekin zehatza izan behar denez 3. ordenako polinomioen oinarria osatzen duten funtzioekin zehatza izatea baino ez da bete behar

Hemendik 4. koefizienteak lor daitezke $\Rightarrow$ $x_1 = \frac{1}{\sqrt{3}}$ $x_2 = -\frac{1}{\sqrt{3}}$

$c_1 = 1$, $c_2 = 1$ $\Rightarrow$ Hauek erabili beste funtzioekin integralak hurbiltzeko:

$$\int_{-1}^{1} f(x)\,dx \simeq f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

- $\int_{-1}^{1} e^x\,dx = 2\sinh(1) = 2'3504$ Metodoa aplikatuz $\Rightarrow$ $\int_{-1}^{1} e^x\,dx \simeq e^{-1/\sqrt{3}} + e^{1/\sqrt{3}} \simeq 2'3427$
  ↳ Analitikoki

Errorea $= \frac{2'3504 - 2'3427}{2'3504} \to \% 0'32$ Bi puntuekin bakarrik $\Rightarrow$ zenbat eta puntu gehiago zehatzeasun handiagoa

17-02-18

**Gauss-en koadratura:** Froga daiteke integrazio formula baldintzak praktikoan jar daitezkeela $x_k$ puntuak polinomio familia jakin batzuen erroak baldin badira:

($f(x)$ eta integrazio tartearen arabera):

| $f(x)$ | Integrazio tartea | ordena bakoitza |
| --- | --- | --- |
| $P_{2n-1}(x)$ | $[-1,1]$ | Legendreren polinomioen erroak |
| $\frac{1}{\sqrt{1-x^2}} P_{2n-1}(x)$ | $[-1,1]$ | Chebysheven polinomioen erroak |
| $x^a e^{-x \cdot p} P_{2n-1}(x)$ | $[0,\infty)$ | Laguerre-ren polinomioen erroak |
| $e^{x^2} P_{2n-1}(x)$ | $(-\infty,\infty)$ | Hermite-ren polinomioen erroak |

Metodoa [-1,1] tartean inplementatu dugu → orduan:

Aldagai aldaketa: $y = \frac{1}{(b-a)}(2x-(a+b))$ eta haren mugak oran -1,1

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{1}{2}(b-a)y + a+b\right)dy$$

* Gauss-en koadratura: formula irekia ⇒ erabili daiteke integral inpropioak ebazteko adibidez ⇒ $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ ez du 0-an ebaluatzen funtzioa

! Gaussen metodoa n puntuekin → 2n-1 ordenako polinomioekin zehatza da

Newton-Côtes metodoa n puntuekin → n-1 ordenako polinomioekin zehatza da

Zehaztasun berdina izateko $n'-1 = 2n-1 \Leftrightarrow n' = 2n$ ⇒ puntu kopuru bikoitza erabili.

17-02-24

Integral inpropioak ebazteko metodoak orain ⇒ Gaussen koadratura (ez dira muturrak ordezkatzen)

Rombergen metodoa ezin da zuzenean aplikatu.

Integral bat inpropioa da baldin eta...

- Integrazio tarteko punturen batean, funtzioaren limitea finitua bada baina ezin bada zuzenean puntu horretan ebaluatu. Ad: $\int_{-1}^{1} dx \frac{\sin x}{x}$ x=0 ezin da ebaluatu
- Integralaren goi eta behe limiteak $a = -\infty$ edo/eta $b = \infty$ baldin badira.
- Integrazio tartean integragarria den singularitate bat agertzen bada.

Adibidez: $\int_0^2 \frac{1}{\sqrt{|x-1|}} dx$ $x=1$ puntuan singularitate bat du; dibergentzia integragarria

∘ Integralaren limiteren baten (a edo/eta b, [a,b] itxiak tartea) singulartasun integragarri bat agertzen badu. Adibidez: $\int_0^1 \frac{1}{\sqrt{x}} dx$

Orokorrean integral inpropio bat ondoko eraten ebaluatu daiteke:

* Tarteoren limiteetan funtzioaren ebaluazioa behar ez den algoritmoaren bat erabili (* Newton-Côtes formula irekiak)

* Aldagai aldaketa bat eginez.

Romberg erabili daiteke baina erdiko punturen edo edozein formula ireki kontuan izanik.

∘ $\int_a^b f(x)\,dx = \int_{1/b}^{1/a} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt \quad ab>0$

Baldin eta $a>0$ eta $b\to\infty$ (edo $a\to-\infty$ eta $b<0$) eta funtzioa $\sim 1/x^2$ txikiagoa bada gutxienez

∘ $\int_a^b f(x)\,dx = \int_0^{\sqrt{b-a}} 2t\, f(a+t^2)\,dt \quad b>a$

Asko jota $(x-a)^{-1/2}$ bezala dibergitzen baldin badu $\sim a$

∘ $\int_a^b f(x)\,dx = \int_0^{\sqrt{b-a}} 2t\, f(b-t^2)\,dt \quad b>a$

Asko jota $(x-b)^{-1/2}$ bezala dibergitzen baldin badu

∘ $\int_0^\infty f(x)\,dx = \int_0^{e^{-a}} f(-\log(t)) \frac{dt}{t}$ Funtzioa esponentzialki txikitzen bada

# Zenbakizko deribazioa:

17-02-28

Algoritmo sinple bat: $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$ $(h \to 0)$

↳ Baina nolakoa?

• Adibidez: $\left.\frac{d(\sin x)}{dx}\right|_{\pi/2} = \cos(\pi/2) = 0$

h txikitzen bagoaz, 0'1-etik adibidez 0-ra gero eta gehiago hurbiltzeko da emaitza. Hala ere, $10^{-4}$-tik aurrera igotzen joango da emaitza. (Baliokide errorea) Zenbakizko deribazioa numerikoki oso ezegonkorra da!

Funtzioa ezaguna badugu beti erabili behar dugu formula analitikoa.

• Erroreak: Bi errore mota ⇒ Deribatua $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$ definitzean arbuiatzen diren terminoengatik (Taylor-en garapena) (serie mozketa: $\varepsilon_t = \frac{h}{2} f''(x_0) + \dots$) (serie motz)

+ byte luzera finitua izateagatik dugun biribilketa. $\left(\varepsilon_r \simeq \left|\frac{f(x_0)}{h}\right| \varepsilon_m\right)$

↓ Makina zehaztasuna

$$f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} \underbrace{- \frac{h}{2} f''(x_0) + \dots}_{\varepsilon_t}$$

$h \to 0$ denean mozketa errorea nulua da baina biribilketa errorea handitzen doa. Orduan konpromezu batera heldu behar gara, bien bitarteko puntu bat hartu behar dugu errore biak minimizatzeko.

Nola minimizatu mozketa/metate errorea? $f(x)$ funtzioaren balioak ezagutuz puntu ezberdinetan $([x_0, f(x_0)], [x_1, f(x_1)], \dots, [x_n, f(x_n)])$ Lagrangeren n ordenako polinomioa kalkulatu: $P_n(x)$ (interpolatu). Orduan hurbilketa:

$$f'(x) \simeq P'_n(x) \quad (n+1 \text{ puntuko formula})$$

Adibidez: 2 puntu: $(x_0, f(x_0))$, $(x_0+h, f(x_0+h))$ ⇒ Datuak

- $P_1(x) = \left(\frac{x_0+h-x}{h}\right) f(x_0) + \frac{x-x_0}{h} f(x_0+h) \Rightarrow P_1'(x) = f'(x) = \frac{f(x_0+h)-f(x_0)}{h} \quad \forall x \in [x_0, x_0+h]$

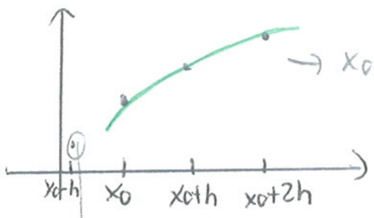
Gure formula ezaguna

3 puntu: $(x_0, f(x_0))$, $(x_0+h, f(x_0+h))$, $(x_0+2h, f(x_0+2h))$

- $P_2(x) = \frac{(x_0+h-x)(x_0+2h-x)}{2h^2} f(x_0) - \frac{(x_0-x)(x_0+2h-x)}{h^2} f(x_0+h) + \frac{(x_0-x)(x_0+h-x)}{2h^2} f(x_0+2h)$

- $P_2'(x) = \frac{x-x_0}{h^2}\left[f(x_0) - 2f(x_0+h) + f(x_0+2h)\right] + \frac{1}{2h}\left(-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right)$

$\forall x \in [x_0, x_0+2h]$




→ $x_0$-tik urruntzen den pausua oso handia denean ez da oso zehatza

→ optimoagoa hau aukeratzen bada: zentratuta badago tartea

$x_0 \to x_0-h$-ra desplazatzen badugu zentratu egingo da tartea $x_0$-n:

$$\begin{cases} f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] & (x_0, x_0+h, x_0+2h) \\ f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] & (x_0-h, x_0, x_0+h) \\ f'(x_0) = \frac{1}{2h}\left[f(x_0-2h) - 4f(x_0-h) + 3f(x_0)\right] & (x_0-2h, x_0-h, x_0) \end{cases}$$

3 puntuko formulak: $O(h^2)$

Askoz hobeto formula hauek 2 puntukoa baino

→ koefiziente guztien batura 0, konstante bati aplikatuz 0 izan behar dutelako

Bigarren deribatua: $P_n(x)$ Lagrangeren $n$ ordenako polinomioa ⇒

$f''(x) \simeq P''_n(x)$ ($n+1$ puntuko formula)

Forward: $f''(x_0) = \frac{1}{h^2}\left[f(x_0+2h) - 2f(x_0+h) + f(x_0)\right]$ — 3 puntuko formula $O(h)$

Centered: $f''(x_0) = \frac{1}{h^2}\left[f(x_0+h) - 2f(x_0) + f(x_0-h)\right]$ — 3 puntuko formula $O(h^2)$

Hau kalkulatzeko: 2. deribatua 1. deribaturen deribatua da, beraz:

$$f''(x_0) = \frac{\frac{f(x_0+h)-f(x_0)}{h} - \frac{f(x_0)-f(x_0-h)}{h}}{h} = \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}$$

Polinomioen interpolaketa egin gabe

# EKUAZIO DIFERENTZIAL ARRUNTAK

17-03-03

$$y'(t) = F(t,y) \quad , \quad t\in[t_a,t_b] \qquad y(t_a)=y_a$$

a) $y$ bektore bat bezela ere ordezkatu dezakegu (ekuazio sistema)

b) $y^n(t) = f(t, y(t), \ldots, y^{n-1}(t))$ ekuazioa era berdinean

$$\begin{cases} y_1'(t) = f_1(t, y_1, y_2, \ldots, y_n) \\ y_2'(t) = f_2(t, y_1, y_2, \ldots, y_n) \\ \vdots \\ y_n'(t) = f_n(t, y_1, y_2, \ldots, y_n) \end{cases}$$

Hasierako baloreen problema:

Mugalde baldintzak

$$y''(t) = F(t, y(t), y'(t)) \qquad y[t_a] = y_a, \; y[t_b] = y_b$$

(askoz konplikatuagoa)

Taylorren garapena:

$$y'(t) = \frac{dy}{dt} = f(t,y) \qquad t_a \le t \le t_b \; , \; y(t_a) = y_a \Rightarrow$$ Taylor-en garapena $\Rightarrow$

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2!} y''(t_i) + \ldots + \frac{h^n}{n!} y^{(n)}(t_i) + O(h^{n+1}) \qquad h = t_{i+1} - t_i$$ Pausua

$$y(t_{i+1}) = y(t_i) + h f(t_i) + \frac{h^2}{2!} f'(t_i) + \ldots + \frac{h^n}{n!} f^{(n-1)}(t_i) + O(h^{n+1})$$

Ezaguna

Eulerren metodoa:

zuzen bat

$$\begin{cases} y(t_1) = y_a \\ y(t_{i+1}) = y(t_i) + h f(t_i) + O(h^2) \end{cases}$$

Lehen ordenako hurbilketa

Adibidez: $y' = y - t^2 + 1$ ⇒ Hemendik $y^{(n)}$ lor daiteke ⇒ Hauek sartu eta hurbilketa zehatzagoa

$$y_{i+1} = y_i + h\,y'(t_i) + \frac{h^2}{2!}\,y''(t_i) + \dots$$

→ Puntu batean $y^{(n)} = y^{(n+1)}$ → esponentziala izango dugu nonbait

$$y'' = y' - 2t = y - t^2 + 1 - 2t\,, \quad y''' = y'' - 2\,, \quad y^{IV} = y''' \dots$$

Ez gara Euler-en hurbilketan geratu behar → Taylorren garapeneko termino gehiago erabili daitezke!

Horrela hainbat puntu lortu eta funtzio hurbildua induktu daiteke

Adibidez: Pendulu ez-lineala. $\theta''(t) + \omega_0^2 \sin(\theta(t)) = 0$ $\quad \theta(t_0) = \theta_0,\ \theta'(t_0) = \Omega_0$

↳ Hurbilketa lineala → $\theta''(t) + \omega_0^2\,\theta(t) = 0$ Angelu txikitan

2. ordeneko ekuazioa 1 ordeneko eta 2 dimentsioko ekuazio bihurtu: $\theta' \equiv y_2\,,\ \theta \equiv y_1$

$$\begin{cases} y_2' = -\omega^2 \sin(y_1) \\ y_1' = y_2 \end{cases} \Rightarrow \vec{y}\,' = \vec{F}(t, \vec{y}) \qquad \left(\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$$

$$\vec{F}(t, \vec{y}) = \begin{pmatrix} y_2 \\ -\omega^2 \sin(y_1) \end{pmatrix}$$

Euler-en metodoa erabili zuzenean.

$$\vec{y}_{i+1} = \vec{y}_i + \vec{F}(\vec{y}_i, t_i)\cdot h$$

17-03-14

Algoritmoa

→ hobeto egingo da Kalkulu eta gehiago kostu izanda

$$\begin{cases} w_0 = y_a \\ w_{i+1} = w_i + h\,T^n(t_i, w_i) \end{cases} \;;\quad T^n(t_i, w_i) = \overset{\text{Taylor}}{y'(t_i)} + \frac{h}{2!}\,y''(t_i) + \dots + \frac{h^{n-1}}{n!}\,y^{(n}(t_i)$$

$y'(t_i) = f(t_i, w_i)$

Baina ahal $y^{(n}$ ez kalkulatzeko beste planteamendu bat.

$$y'(x) = \frac{dy(x)}{dx} = f(x, y(x))\,; \quad x_a \le x \le x_b\,, \quad y(x_a) = y_a$$

Ordena handiko metodoa: nola eliminatu deribatuen kalkulua?

Adibidez: $w_0 = y_a$

$$w_{i+1} = w_i + hT^{(2)}(x_i, w_i)$$

$$T^{(2)}(x_i, w_i) = f(x_i, w_i) + \frac{h}{2} f'(x_i, w_i) + O(h^2)$$

$$T^{(2)} \equiv f(x,y) + \frac{h}{2}\left(\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} f(x,y)\right) \qquad (1)$$

## Runge-Kutta metodoa: (RK) (Adibide hau RK2 da)

"Simulatu" daiteke $T^{(2)}$ funtzioa puntu berezi batzuetan ebaluatutako $f$ funtzioaren balioekin? $T^{(2)} \approx a_1 f(x+\alpha_1, y+\beta_1)$ ?

desplazatu t eta y-n

$$a_1 f(x+\alpha_1, y+\beta_1) \approx a_1\left[f(x,y) + \alpha_1 \frac{\partial f(x,y)}{\partial x} + \beta_1 \frac{\partial f(x,y)}{\partial y}\right] \qquad (2)$$

Taylor

(1) eta (2) berdinduz: $a_1 f(x,y) + a_1 \alpha_1 \frac{\partial f(x,y)}{\partial x} + \beta_1 \frac{\partial f(x,y)}{\partial y} a_1 =$

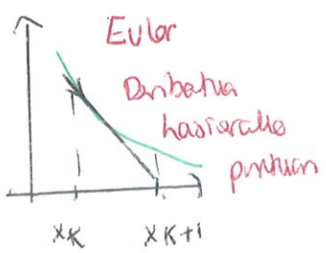
$$f(x,y) + \frac{h}{2}\frac{\partial f(x,y)}{\partial x} + \frac{h}{2}\frac{\partial f(x,y)}{\partial y} f(x,y) \Rightarrow a_1 = 1, \ \alpha_1 = \frac{h}{2}, \ \beta_1 = \frac{h}{2} f(x,y)$$

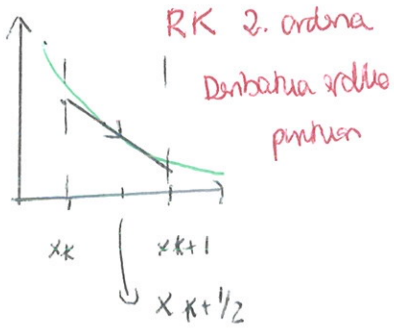
Orduan $T^{(2)}$ ordez idatzi $T^{(2)} \approx f\left(x + \frac{h}{2}, y + \frac{h}{2} f(x,y)\right)$

2. ordenako Taylorren hurbilpena erabili daiteke bigarren deribatua esplizituki kalkulatu gabe.

Beraz $\Rightarrow$
$$\begin{cases} w_0 = y_0 \\ w_{i+1} = w_i + h f\left(x_i + \frac{h}{2}, w_i + \frac{h}{2} f(x_i, w_i)\right) \end{cases} \quad i = 0, \dots, N-1$$

$N+1 \equiv$ puntu kopurua




Zehatzagoa deribatua erdiko puntuan egiten bada.

17-03-17

## Runge-Kutta 4:

- $\dot{y}(t) = \frac{dy(t)}{dt} = f(t, y(t))$ , $t_a \leq y \leq t_b$ , $y(t_a) = y_a$

$$\begin{cases} w_0 \equiv y_a \ , \quad w_{i+1} = w_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \quad i = 0, \dots, N-1 \\ K_1 \equiv hf(t_i, w_i) \ , \quad K_2 \equiv hf(t_i + \frac{h}{2}, w_i + \frac{K_1}{2}) \ , \quad K_3 \equiv hf(t_i + \frac{h}{2}, w_i + \frac{K_2}{2}) \ , \\ K_4 \equiv hf(t_i + h, w_i + K_3) \end{cases}$$

# SISTEMA LINEALAK: autobalio / bektore problema

17-03-28

Schrödingeren ekuazio geldikorra aztertuko dugu diferentzia finituak erabiliz:

$$\underbrace{\left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right)}_{\hat{H}} \phi_n(x) = E_n \phi_n(x) \qquad \left\{ \begin{array}{l} \hbar = 1 \\ m = 1 \\ e = 1 \end{array} \right. \quad a.u$$

Problema honetan potentziala emanda dagoela suposatuko dugu eta uhin-funtzioak $(\phi_n(x))$ eta autobalioak $(E_n)$ aurkitu behar ditugu emandako potentziala batentzat, $V(x)$.

$\hat{H}$ Hamiltondar eragilea matrize eragile bezala bihurtu behar dugu.

$$\hat{H} = \left(-\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right) \longrightarrow H_m$$

Horretarako, espazioa ez diskretoan kontsideratuko dugu. Demagun $[a, b]$ tarte bat aztertu nahi dugula, eta bertan $N$ puntu ditugula.

$$x_i = a + \frac{(b-a)}{N-1}(i-1) \ , \quad i = 1, \dots, N \qquad \left(h = \frac{b-a}{N-1}\right)$$

Era berean, lortu nahi diren autofuntzioak puntu horietan ebaluatutako autobektoreekin ordenatuko dira $\phi_n(x) \rightarrow \{\Phi_n^i\} = \begin{pmatrix} \Phi_n^1 \\ \Phi_n^2 \\ \vdots \\ \Phi_n^N \end{pmatrix}$

Bestalde, bigarren deribatu eragilea hurrengo eran idatzi daiteke:

$$\frac{d^2}{dx^2} \rightarrow \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}$$

→ 2 deribatuaren forma lortzen delako horrela

$$\frac{\phi_n^{i-1} - 2\phi_n^i + \phi_n^{i+1}}{h^2}$$

Potentziala : $V(x) \rightarrow \begin{pmatrix} V(x_1) & 0 & \cdots & & 0 \\ \vdots & V(x_2) & & & \vdots \\ \vdots & & V(x_3) & & \vdots \\ 0 & \cdots & \cdots & \cdots & V(x_N) \end{pmatrix}$

Beraz →

$$\begin{pmatrix} \frac{1}{h^2}+V(x_1) & -\frac{1}{2h^2} & 0 & 0 & \cdots & 0 \\ -\frac{1}{2h^2} & \frac{1}{h^2}+V(x_2) & -\frac{1}{2h^2} & \cdots & & 0 \\ \vdots & & & \ddots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & -\frac{1}{h^2}+V(x_N) \end{pmatrix}\begin{pmatrix} \Phi_n^1 \\ \Phi_n^2 \\ \vdots \\ \Phi_n^N \end{pmatrix} = E_n \begin{pmatrix} \Phi_n^1 \\ \Phi_n^2 \\ \vdots \\ \Phi_n^N \end{pmatrix}$$

Balio autobalitzaren problema

↳ Matrize tridiagonala

Puntu asko kontsideratzen h distantzia txikitu eta uhin-funtzio gehiago agertuko dira (haliz eta oso antzekoak izan haien artean)

Horrela $\Phi_n^i$-ak lortuko ditugu eta horrela modulatuz uhin-funtzioa (autobalitzaea) lortuko dugu.

17-03-30

## Ekuazio sistema linealen zenbakizko soluzioa.

Inplementazio ezberdinak (metodo ezberdinak), matrizeen mota batzuen propietate bereziak esplotatzen dituzte, azkarrago, zehatzago eta memoria (RAM) minimoa kontsumitzeko. Aparteko

## Mugalde-problemak:

Tiroaren metodoa:

Dirichlet-en MB

$$y''(x) = f(x,y,y') \qquad y(a)=y_a\,,\quad y(b)=y_b \qquad \text{(Mugalde baldintzak)}$$

Hasierako-baldintzen problema batera hurbildu: $y'(a)=\alpha \Rightarrow$ parametro bat

Helburua $\Rightarrow$ ekuazioa ebaztea eta $y_\alpha(b)=y_b \Rightarrow y(b,\alpha)-y_b$

alfaren menpekoa izango da

funtzioaren erroak aurkitu

- Newton
- Erdibiketa
- Sekantearen metodoa

$\alpha$-ren balio batekin hasi eta balio ezberdinak ematen joan metodo horren bidez $|y_\alpha(b)-y_b| \le \varepsilon$ izan arte.

Newton: $\alpha_0$ batekin hasi eta $\alpha_{k+1}=\alpha_k - \dfrac{y(b,\alpha_k)-y_b}{\left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=\alpha_k}}$ $\quad k\geq 0$

Arazoa $\Rightarrow$ $\left(\frac{\partial y}{\partial \alpha}\right)$ ezezaguna da: $y''=f(x,y(x,\alpha),y'(x,\alpha))$

$$\frac{\partial y''}{\partial \alpha} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}+\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}+\frac{\partial f}{\partial x}\cancel{\frac{\partial x}{\partial \alpha}} \Rightarrow \text{Aldagai aldaketa: } z(x,\alpha)=\frac{\partial(y(x,\alpha))}{\partial \alpha}$$

$z(a,\alpha)=\dfrac{\partial(y(a,\alpha))}{\partial\alpha}=\dfrac{\partial(y_a)}{\partial\alpha}=0$

$\dfrac{d}{dx}\left(\dfrac{\partial(y(x,\alpha))}{\partial\alpha}\right)=\dfrac{\partial}{\partial\alpha}y'(x,\alpha)$

x-rekiko

$$z''=\frac{\partial f}{\partial y}z+\frac{\partial f}{\partial y'}z' \qquad z(a,\alpha)=0\,,\quad z'(a,\alpha)=1$$

$\frac{\partial}{\partial\alpha}$ deribatu x independentea da

Hemendik 4 dimentsioko bektore bat: $u=\begin{pmatrix} y(x,\alpha)\\ y'(x,\alpha)\\ z(x,\alpha)\\ z'(x,\alpha)\end{pmatrix} \rightarrow u'=\begin{pmatrix} u_2\\ f(x,u_1,u_2)\\ u_4\\ \frac{\partial f}{\partial y}u_3+\frac{\partial f}{\partial y'}u_4\end{pmatrix}$

$u(a)=\begin{pmatrix} y_a\\ \alpha\\ 0\\ 1\end{pmatrix} \Rightarrow$ rk4 aplikatu.

Lortuko den $u_1=y(x,\alpha)$-rekin $\Rightarrow \alpha_{k+1}=\alpha_k-\dfrac{u_1(b,\alpha_k)-y_b}{u_3(b,\alpha_k)}$

Amaitu $|y(b,\alpha)-y_b|\le\varepsilon$ denean.

Diferentzia finituak.

Deribatuak erdiko puntuko diferentzia finituekin ordezkatu:

Mutineko sarrera balioak finituak $\Leftarrow$

$$y'(x_i) = \frac{1}{2h}\left(y(x_{i+1}) - y(x_{i-1})\right)$$

$\rightarrow$ Taulako mugetan aplikatu behar badira $\Rightarrow$ forward/backward erabili

$$y''(x_i) = \frac{1}{h^2}\left(y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right)$$

$\rightarrow$ y ez dugu ezagutzen $\rightarrow$ hau gutxiago puntu ezberdinetan ebaluatuta $(x_{i+1}, x_i, x_{i-1})$

y ezagutzen ez dugunez ezin izango dugu zuzenean ordezkatu eta ekuazioa ebatzi $\Rightarrow$ hurbilketa bat hartu hasieran $y_0(x)$ $\rightarrow$ 0. a $\rightarrow$ hobetzen joango gara

Haren inguruan $y''(x_i) = f(x_i, y(x_i), y'(x_i))$-ren Taylor-en hurbilketa egin.

- $y''(x_i) = f(x_i, y(x_i), y'(x_i)) \overset{\text{linealizatu}}{\simeq} f(x_i, y_0(x_i), y'_0(x_i)) + \left.\frac{\partial f}{\partial y}\right|_{x_i, y_0, y_0'} (y(x_i) - y_0(x_i)) +$

$\left.\frac{\partial f}{\partial y'}\right|_{x_i, y_0, y'_0} (y'(x_i) - y'_0(x_i)) + \dots$ $\rightarrow$ lineala bada hau zuzenean dugu: $y'' + Q(x)y' + P(x)y = F(x)$

$\rightarrow$ y berria $\rightarrow$ $y_1(x)$

- Orain hurbildu $y''(x_i) = \dfrac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}$ $\Rightarrow$ Ekuazio lineala lortu.

$$y_1(x_{i+1}) - 2y_1(x_i) + y_1(x_{i-1}) = h^2\Big[\underbrace{f(x_i, y_0(x_i) + y'_0(x_i))}_{F_0(x_i)} + Q_0(x_i, y_0, y_0')\,(y_1(x_i) - y_0(x_i)) +$$

$$P_0(x_i, y_0, y_0')\left[\frac{1}{2h}(y_1(x_{i+1}) - y_1(x_{i-1})) - y'_0(x_i)\right]\Big] \Rightarrow \text{Askatu } y_1$$

termino guztiak $y_0$ termino ezagunen menpe. EZEZAGUNAK

$\rightarrow$ Hau behin eta berriz aplikatu konbergentzia ~~lortzeko~~

$$y_1(x_{i+1}) - 2y_1(x_i) + y_1(x_{i-1}) - h^2 Q_0(x_i)\, y_1(x_i) - \frac{h}{2} P_0(x_i)\,(y_1(x_{i+1}) - y_1(x_{i-1})) =$$

$$h^2 F_0(x_i) - Q_0(x_i)\, y_0(x_i)\, h^2 - \frac{h}{2} P_0(x_i)\, y_0'(x_i)$$ $\rightarrow$ EZAGUNAK (dena $x_i$-ren menpe) Matrize tridiagonala

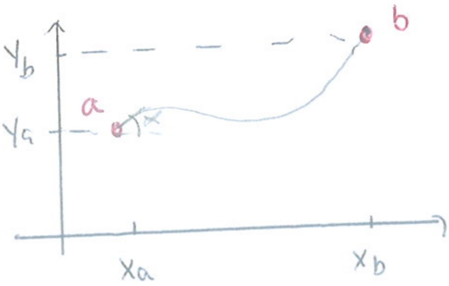
$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1+\frac{h}{2}P_0(x_2) & -2-h^2Q_0(x_2) & 1-\frac{h}{2}P_0(x_2) & 0 & \dots \\ 0 & 1+\frac{h}{2}P_0(x_3) & -2-h^2Q_0(x_3) & 1-\frac{h}{2}P_0(x_2) & \dots \\ \vdots & 0 & \vdots & \ddots & \\ \vdots & \vdots & \vdots & \dots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_a \\ h^2F_0(x_2) - Q_0(x_2)\,y_0(x_2)\,h^2 - \frac{h}{2}P_0(x_2)\,y_0'(x_2) \\ \vdots \\ \vdots \\ y_b \end{pmatrix}$$

## Ekuazio diferentzial arruntak: Muga baloreen problema.

↗ Mugalde baldintzak

$$y''(x) = \frac{d^2y(x)}{dx^2} = f(x, y(x), y'(x)) \quad , \quad y(x_a)=y_a \, , \quad y(x_b)=y_b$$

"Tiro"-aren metodoa. ↱ a→b-ra doan funtzioa $y''(x)=f(x,y,y')$ betez. → "proba-errore"




• $y'(x_a)=\alpha$ suposatu (hasierako malda) ⇒ $x_b$ puntuan $y(x_b,\alpha)$ izango dugu ($\alpha$-ren menpekoa) ⇒ Helburua: $y(x_b,\alpha)-y_b=0$ lortzea eta $\alpha$ lortzean soluzioa lortuko da.

* Hau lortzeko hasierako baloreen problema ebatzi, $y(x_a)=y_a$ eta $y'(x_a)=\alpha$-rekin.

• $y(x_b,\alpha)-y_b=0$ ebaztea zaila izan daiteke analitikoki ⇒ numerikoki ebatz daiteke Newton, Sekante edo Erdibiketa metodoa erabiliz.

Gerta daiteke 0, 1 edo $\infty$ soluzio izatea!

17-04-03

## DERIBATU PARTZIALETAKO EKUAZIOAK:

* Difusio ekuazioa: $\frac{\partial\Phi(t,x,y,z)}{\partial t} = K\nabla^2\Phi(t,x,y,z)$

• $\boxed{J=-K\,\mathrm{grad}(\rho)}$ → "We know "from experience" that a high concentration induces a "diffusion current" towards the direction where concentration is lower" Fick-en legea.


$$\frac{\partial\rho}{\partial t}=K\nabla^2\rho \leftarrow \frac{\partial\rho}{\partial t} = -\mathrm{div}(J)$$

↳ Karguaren kontserbazio legea

(jarraitasun ekuazioa)

Ebazteko ⇒ 1.D-ko problema: $\Phi_{i-1,n} = \Phi(x_i, t_n)$ , $x_i = i \cdot h$ , $t_n = \tau \cdot n$

$h \to$ pausua x-n, $\tau \to$ pausua t-n.

Metodo esplizitua:

$$* \quad \frac{\Phi_{i,n+1} - \Phi_{i,n}}{\tau} = K \frac{\Phi_{i+1,n} - 2\Phi_{i,n} + \Phi_{i-1,n}}{h^2} \to \left( \frac{\partial \Phi}{\partial t} = K \frac{\partial^2 \Phi}{\partial x^2} \right)$$

$$* \quad \Phi_{i,n+1} = \Phi_{i,n} + \underbrace{(\tau K / h^2)}_{= r} (\Phi_{i+1,n} - 2\Phi_{i,n} + \Phi_{i-1,n})$$

metodoa inplizitua doa

↳ diskretizazioaren menpekoa ($r > \frac{1}{2} \to$ ezegonkorra!)

Honekin batera hasierako balioa eta mugako baldintzak behar ditugu:

* $\Phi(L,t) = g_B(t)$, $\Phi(0,t) = g_A(t)$ (Dirichlete, baina Neumann ere)

Metodo esplizitua deitzen da aurreko puntuen rahikoa dekako hurrengoak ezagutzeko.

Metodo inplizitua: Crank-Nicholson metodoa ⇒ (metodo hobea)

$$\frac{\Phi_{i,n+1} - \Phi_{i,n}}{\tau} = \frac{\Phi_{i+1,n} - 2\Phi_{i,n} + \Phi_{i-1,n}}{h^2} K/2 + \frac{\Phi_{i+1,n+1} - 2\Phi_{i,n+1} + \Phi_{i-1,n+1}}{h^2} K/2$$

↳ aurreko puntuaren lotuk eta ez hasierakoa

$$r = K\tau/h^2 \Rightarrow \underbrace{-r\,\Phi_{i-1,n+1} + 2(1+r)\Phi_{i,n+1} - r\,\Phi_{i+1,n+1}}_{\text{ezezagunak}} = \underbrace{r\,\Phi_{i-1,n} + 2(1-r)\Phi_{i,n} + r\,\Phi_{i+1,n}}_{\text{ezagunak}}$$

↳ ez du balore kritikorik

* Ekuazio sistema planteatu: (tridiagonala)

$$\begin{pmatrix} \ddots & \ddots & \\ -r & 2(1+r) & -r \\ & \ddots & \ddots \end{pmatrix} \underbrace{\begin{pmatrix} \Phi_{1,n+1} \\ \vdots \\ \Phi_{i,n+1} \\ \vdots \\ \Phi_{N,n+1} \end{pmatrix}}_{\text{ezezaguna}} = \underbrace{\begin{pmatrix} \ddots & \ddots & \\ r & 2(1-r) & r \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \Phi_{1,n} \\ \vdots \\ \Phi_{i,n} \\ \vdots \\ \Phi_{N,n} \end{pmatrix}}_{\text{ezaguna}} = \begin{pmatrix} B(0) \\ \vdots \\ B(i) \\ \vdots \\ B(N) \end{pmatrix}$$

Metodo hau erabili daiteke Schrödingeren ekuazioa ebazteko ⇒ ezberdintasun bakarra r konplexua izango dela da.

17-04-11

$m, \hbar = 1$

Schrödingeren ekuazioa: $$i\frac{\partial\Psi(\vec{r},t)}{\partial t} = -\frac{1}{2}\nabla^2\Psi(\vec{r},t) + V(\vec{r},t)\Psi(\vec{r},t), \quad \Psi(\vec{r},0)=\Psi_0(\vec{r})$$

Dimentsio batean; 1D $\Rightarrow$ $$i\frac{\partial\Psi(x,t)}{\partial t} = -\frac{1}{2}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t)$$

Diskretizatu $\rightarrow$
$$\begin{cases} x \Rightarrow x_i = ih & (\text{pausua} \rightarrow h) \\ t \Rightarrow t_n = n\cdot\tau & (\text{pausu denborala} \rightarrow \tau) \end{cases}$$

* Metodo esplizitua $\rightarrow$ $\Psi^n = (\Psi_1^n \Psi_2^n \ldots \Psi_N^n)$, $H^n = H^n{}_{i,j} = \delta_{i,j}(V_i^n + 1/h^2) +$

$$-\frac{1}{2}\delta_{i,j+1}/h^2 - \frac{1}{2}\delta_{i+1,j}/h^2 \Rightarrow i\frac{\Psi_i^{n+1}-\Psi_i^n}{\tau} = \frac{-\Psi_{i+1}^n + 2\Psi_i^n - \Psi_{i-1}^n}{2h^2} + V_i^n\Psi_i^n$$

$$H\Psi = i\frac{\partial\Psi}{\partial t}$$

$$i\frac{\Psi^{n+1}-\Psi^n}{\tau} = H^n\Psi^n \Rightarrow \Psi^{n+1} = \Psi^n - i\tau H^n\Psi^n = (\mathbb{1} - i\tau H^n)\Psi^n$$

Metodo nahiko kaskarra

* Metodo inplizitua (Crank–Nicolson, 1D) $\rightarrow$ $$i\frac{\Psi^{n+1}-\Psi^n}{\tau} = \frac{1}{2}\left[H^n\Psi^n + H^{n+1}\Psi^{n+1}\right]$$

$$\Psi^{n+1} = \underbrace{\left(\mathbb{1} + \frac{i\tau}{2}H^{n+1}\right)^{-1}\left(\mathbb{1} - \frac{i\tau}{2}H^n\right)}_{\text{ezaguna}}\Psi^n = F\Psi^n$$

$\hookrightarrow$ ezezaguna

Bi aukera posible:

$\rightarrow$ V t-ren independentea denean

1) Zuzenean F matrizearen alderantzizkoa kalkulatu eta hurrengo iterazioa kontsideratu:

$$\Psi^{n+1} = F\Psi^n$$

$\rightarrow H^{n+1} = H^n$

2.) Alderantzizkoa zuzenean kalkulatu behaerean sistema hiru-diagonal bat askatu $\Rightarrow$ $$\Psi^{n+1} = \left(\mathbb{1} + \frac{i\tau}{2}H^n\right)^{-1}\left(\mathbb{1} - \frac{i\tau}{2}H^n\right)\Psi^n = \left(\mathbb{1}+\frac{i\tau}{2}H^n\right)^{-1}\left(2\mathbb{1} - \left(\mathbb{1}+\frac{i\tau}{2}H^n\right)\right)\Psi^n =$$

$$(Q^{-1} - \mathbb{1})\Psi^n, \quad Q = \frac{1}{2}\left(\mathbb{1} + \frac{i\tau}{2}H^n\right) \Rightarrow \Psi^{n+1} = Q^{-1}\Psi^n - \Psi^n = \boxed{\Xi}^n - \Psi^n$$

$\hookrightarrow Q\Xi^n = \Psi^n$

* Metodo espektrala: $\Psi(\vec{r},0) = \Psi_0(\vec{r})$

Lehenengo ekuazio geldikorra kontsideratu ($V$ $t$-ren independentea izanik) egin behar da. Hauek dira egoera eta energia autobalioak:

$$-\frac{1}{2}\nabla^2 \Phi_j(\vec{r}) + V(\vec{r})\,\Phi_j(\vec{r}) = \varepsilon_j\,\Phi_j(\vec{r})$$

Beraz $\Rightarrow$ $\Psi(\vec{r},t) = \sum_j e^{-i\varepsilon_j t}\,\Phi_j(\vec{r})\,A_j$ , $A_j = \int dr\, \Phi_j(r)^* \,\Psi_0(r)$

Inder bat aplikatzen soka batean: $T\frac{\partial^2\phi(x,t)}{\partial x^2} + F(x,t) = \mu\frac{\partial^2\phi(x,t)}{\partial t^2}$

17-05-02

Uhin-ekuazioa: Presio uhina, soinua, ingurune elastiko propagazioa

$$\frac{\partial^2 \rho(\vec{r},t)}{\partial t^2} = v^2\nabla^2\rho(\vec{r},t) \qquad \begin{cases} \rho(\vec{r},t=0) = f(\vec{r}) \\ \frac{d\rho}{dt}(\vec{r},t=0) = g(\vec{r}) \end{cases}$$

1D $\Rightarrow$ $$\frac{\partial^2\rho(x,t)}{\partial t^2} = v^2\frac{\partial^2\rho(x,t)}{\partial x^2} \;;\quad \rho(x,t=0) = f(x),\quad \frac{d\rho}{dt}(x,t=0) = g(x)$$

$$\frac{\rho_i^{n+1} + \rho_i^{n-1} - 2\rho_i^n}{\tau^2} = v^2\,\frac{\rho_{i+1}^n + \rho_{i-1}^n - 2\rho_i^n}{h^2}$$

$\leq 1$

Eskema esplizitua $\Rightarrow$ $$\rho_i^{n+1} = 2\rho_i^n - \rho_i^{n-1} + \left(\frac{v\tau}{h}\right)^2\left(\rho_{i+1}^n + \rho_{i-1}^n - 2\rho_i^n\right)$$

Hasierako baldintzak: $\rho_i^{n=1} = f_i$ , $\rho_i^{n=2} - \rho_i^{n=1} = \tau\cdot g_i$

2D $\Rightarrow$ $$\frac{\partial^2\rho(x,t,y)}{\partial t^2} = v^2\left(\frac{\partial^2\rho(x,y,t)}{\partial x^2} + \frac{\partial^2\rho(x,y,t)}{\partial y^2}\right) \;;\quad \rho(x,y,t=0) = f(x,y)$$

$$\frac{d\rho}{dt}(x,y,t=0) = g(x,y)$$

Mugalde baldintzak

- $\rho_{i,j}^{n=1} = f_{i,j}$ ; $\dfrac{\rho_{i,j}^{n=2} - \rho_{i,j}^{n=1}}{\tau} = g_{i,j}$  Mugalde-baldintzak.

- $$\rho_{i,j}^{n+1} = 2\rho_{i,j}^n - \rho_{i,j}^{n-1} + \left(\frac{v_x\tau}{h_x}\right)^2\left(\rho_{i+1,j}^n + \rho_{i-1,j}^n - 2\rho_{i,j}^n\right) + \left(\underbrace{\frac{v_y\tau}{h_y}}\right)^2\left(\rho_{i,j+1}^n + \rho_{i,j-1}^n - 2\rho_{i,j}^n\right)$$

baldintza kritikoa $< 1/2$

Laplace-ren ekuazioa 2D-n: $\nabla^2\phi(r)=0$

Difusio ekuazioaren egoera egonkorra

Difusio-ekuazioaren kasu partikulartzat har daiteke, non $\dfrac{\partial\Phi(r,t)}{\partial t}=0$

- $$\nabla^2\Phi(r)=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=\frac{\Phi_{i+1,j}^n-2\Phi_{i,j}^n+\Phi_{i-1,j}^n}{h^2}+\frac{\Phi_{i,j+1}^n-2\Phi_{i,j}^n+\Phi_{i,j-1}^n}{h^2}=0$$

  2D-n

$$\Rightarrow \Phi_{i+1,j}^n-2\Phi_{i,j}^n+\Phi_{i-1,j}^n+\Phi_{i,j+1}^n-2\Phi_{i,j}^n+\Phi_{i,j-1}^n=0\rightarrow$$

$$\Phi_{i+1,j}^n+\Phi_{i-1,j}^n+\Phi_{i,j+1}^n+\Phi_{i,j-1}^n=4\Phi_{i,j}^n$$

- $$\frac{\partial\Phi(x,y,t)}{\partial t}=\frac{\Phi_{i,j}^{n+1}-\Phi_{i,j}^n}{\tau}=0\rightarrow\Phi_{i,j}^{n+1}=\Phi_{i,j}^n$$

Beraz $\Rightarrow$ $$\Phi_{i,j}^{n+1}=\frac{1}{4}\left(\Phi_{i+1,j}^n+\Phi_{i-1,j}^n+\Phi_{i,j+1}^n+\Phi_{i,j-1}^n\right)$$

hasierako suposizio bat
$\Phi(x,y,t=0)=f(x,y)$
zenbat $\Rightarrow$ hobeto joan

Laplace-n ekuazioan denbora ez da esplizituki agertzen baina difusio ekuazioa bezala aztertu daiteke denbora aldagai artifizial bat gehituz eta soilik egoera egonkorra interesatzen zaigunez (iterazio asko pasata) Horregatik, denbora pausu posibleen arteko handiena hartzea komeni da: $r=1/4$

Hau askotan erabiltzen da $\Rightarrow$ denbora artifizialki sortu eta iterazio asko pasatzean hasi daitezke hartzen

# MONTE CARLO METODOA:

17-05-05

- Zorizko zenbaki multzo handiak erabiltzean oinarrituta dauden metodoak dira.
- Integralak ebazteko metodoa.
- Aplikazioak gaur egun:
  - Fisika estatistikoa
  - Gorputz askoren mekanika kuantikoa
  - Matematika aplikatua
  - Jokabide sozialen / biologikoen azterketa
  - Erradiazio garraio problemak
  - Trafiko simulazioak.

Adibidez: $\pi = 2\int_{-1}^{1}\sqrt{1-x^2}\,dx \simeq \frac{2}{N}\cdot 2\sum_{i=1}^{N}\sqrt{1-x_i^2}$ ( $\{x_i\} \to [-1,1]$ tarteko zorizko zenbakien multzoa)

Orduan $\Rightarrow \int g(R)\, f(R)\, d^N R \simeq \frac{1}{N}\sum_{i=1}^{N} g(\psi_i)$ $\psi_i \Rightarrow f(R)$ banaketa jarraitzen duen zenbaki aleatorioa

Dimentsio handiko integralak kalkulatzeko diseinatutako metodoa da.

Adibidez $\Rightarrow$ $G = \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\cos(x)\,dx$

↳ distribuzio gaussiarra $\Rightarrow f(x) = \frac{e^{-x^2/\sigma^2}}{\sigma\sqrt{\pi}}$

Distribuzioa jarraitzen duen $\psi$ aldagai aleatorioa sortuz, integrala honela era estatistikoan kalkulatu daiteke:

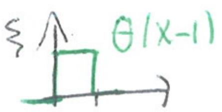
$$G \simeq \frac{1}{N}\sum_{i=1}^{N}\cos(\Psi_i)$$

Azkenean $\Rightarrow$ probabilitate handiena duten horietan ebaluatu. ("importance sampling")

ez banaketa homogeneoa

↳ puntu batzuetan besteetan baino probabilitate handiagoa dutelako

Arazoa $\Rightarrow$ probabilitate hori jarraitzen duen lagina sortu.

Fortran-en 

Banaketa lortzen dugu call random-number-ekin

Baina $f(x)$  nahi badugu?

Ekuazio hau askatu $\Rightarrow$ $\int_{-\infty}^{\Psi} f(x)\,dx = \xi$ $\Rightarrow$ $\Psi(\xi)$ lortu.

Metodoa hau lortzeko $\Rightarrow$ Metropolis algoritmoa

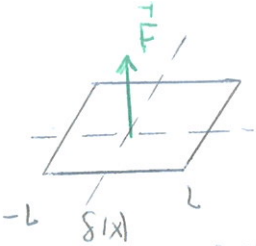
Adibidez $\Rightarrow$ Demagun $n=100$ partikula ditugula $\Rightarrow$ sistemak $N=3n$ dimentsio ditu.

Sistemaren mikroegoera deskribatzen duen bektorea: $\vec{R}_i = (R_1, R_2, \dots, R_N)$

Honekin $E_i$ bat izango dugu lotuta $\Rightarrow$ $P_i = e^{-E_i/k_B T} / Z$ probabilitatea

Anketa: (Aztertuta) 2014ko uztaila

2017-04-28




Denbor bat, muga osoan desplazamendua nulua da, $\phi = 0$

$$c^2 \sigma^2 \left[ \underbrace{\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2}}_{\nabla^2 \phi(x,y)} \right] = F \delta(x) \delta(y)$$ (Poisson-en ekuazioa)

emaitza honetara egonkortuko da difusio-ekuazioa

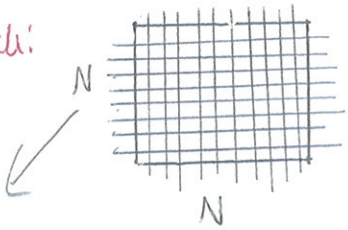
$$\begin{cases} \nabla^2 \phi(x,y) = 0 \rightarrow K \nabla^2 \phi(x,y) = \dfrac{\partial \phi(x,y)}{\partial t} \\ \nabla^2 \phi(x,y) = \rho(x,y) = \dfrac{1}{c^2 \sigma^2} F \delta(x) \delta(y) \end{cases}$$

$t$ eta $K$ guk asmatuko ditugu ; $t \to \infty$ denean egonkortzen da. (Difusio ekuazio bihurtu)

$$\hookrightarrow K \left\{ \nabla^2 \phi(x,y) - \rho(x,y) \right\} = \frac{\partial \phi(x,y)}{\partial t}$$

$$K \left\{ \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{h^2} - \rho_{ij} \right\} = \frac{\phi_{ij}^{n+1} - \phi_{ij}^{n}}{\tau}$$

Mugalde baldintzak:




N zatitan banatu

Denbora artifizialki sartu dugunez hasierako egoera asmatuko dugu: $\begin{cases} n=1 \\ t=0 \end{cases} \Rightarrow \phi_{ij}^{n=1} = 0$

Denbora gutxia luna

do $i = 2, N-1$

do $j = 2, N-1$

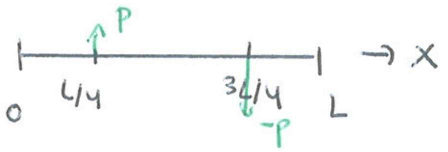
$\nabla^2 \phi = \rho$

$$\phi_{berria}(i,j) = \frac{\tau K}{h^2} \left\{ \nabla^2 \phi_{zaharra} - h^2 \rho_{ij} \right\} + \phi_{zaharra}$$

$\hookrightarrow \phi_{i,j}^{n+1}$

$1/4$ -ari

Hagaxkaren anketa:




$u(x) \rightarrow x$ puntuko anplitudea

Lehenengo partea $\Rightarrow$ sokaren itxura kalkulatzea denbora luzez $p$ aplikatzean.

Sokaren dinamika $\rightarrow$ $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ ($\sim m \cdot a = F$)

$\searrow$ indarra (azelera unitateko edo dentsitatea)

Indar hori sokaren kurbadurak sortutako tentsioarekin lotuta dago. Dena dela indar bertsalu bat egon daiteke (presio bat aplikatzean besteak beste). Beraz:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t)$$

dena delako dimentsioetan

Gure indarra $t$-ren independentea $\Rightarrow$ $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + p(x)$

denbora asko itxarotean (egonkortzen da forma)

$u$ denboraren independentea izango da $\Rightarrow$ $c^2 \frac{\partial^2 u}{\partial x^2} + p(x) = 0$

Ebazteko modu bat:

$i=1$ —— $i=N$

muturrak geldi

$i \neq 1, N \Rightarrow \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + p_i = 0$, $u_1 = 0$, $u_N = 0$

sistema tridiagonala ebatzi

Beste metodoa (ordenera ederen dimentsionalak):

$\frac{\partial^2 u}{\partial x^2} + p(x) = 0 \rightarrow$ $t$ artifizial bat kontsideratu difusio ekuazio bihurtzeko (difusio-ekuazioan (balantzea) $t$ handia denean amaiera denboraren independentea delako) $\Rightarrow$ denbora asko pasatzean hori izango da gure soluzioa.

$$K\left(\frac{\partial^2 u}{\partial x^2} + p(x)\right) = \frac{\partial u(x,t)}{\partial t}$$

($c=1$ jarri)

Hau ebazteko hasierako ekuazio bat behar dugu, berdin digu zein,

beti lortuko dugu soluzioa

$u_i^{n=0} = 0$ hartu adibidez.

$K$ edozein izan daiteke $r = \frac{\tau K}{h^2}$ egonkorra den bitartean $\rightarrow r = \frac{1}{4}$

hartu (kontuz) $\Rightarrow$ $\frac{u_i^{n+1} - u_i^n}{\tau} = K\left\{\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + p_i\right\}$

hauxe

geldi dago

Gero presioa erasiten uzten denean uhin-ekuazioa erabili hasierako egoera eta $v=0$.

# FISIKA KUANTIKOA 2. KUATRIA, 2. PARTEA

17-03-27

## Partikula bereiztezinak eta atomo elektroianitzak

- Demagun bi partikula ditugula eta uhin-funtzioa hauxe dela:

$$\psi(x_1,x_2) = \psi_a(x_1)\,\psi_b(x_2)$$

non $\psi_a$ eta $\psi_b$ ortogonalak diren. → ez dago elkarreraginik.

*Suposa dezagun gainera bosoiak direla → simetrikoa ⇒ simetrizatu behako dugu:

$$\psi_B(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)\right]$$

*Suposa dezagun fermioiak direla ⇒ antisimetrikoa:

$$\psi_F(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)\right]$$

Ariketa: Zein da partikulen arteko batez besteko distantzia? Zein da ezberdintasuna partikulak bosoiak edo fermioiak direnean?

$d = |x_1 - x_2|$ ? ⇒ $\langle (x_1-x_2)^2 \rangle$ kalkulatu → $d = \sqrt{\langle (x_1-x_2)^2\rangle}$

$$\langle (x_1-x_2)^2\rangle_\psi = \langle x_1^2 + x_2^2 - 2x_1x_2\rangle_\psi = \langle x_1^2\rangle_\psi + \langle x_2^2\rangle_\psi - 2\langle x_1x_2\rangle_\psi$$

$$* \ \langle x_1^2\rangle_\psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2\,\psi(x_1,x_2)\,\psi^*(x_1,x_2)\,dx_1\,dx_2 = \int_{-\infty}^{\infty}\psi_a^*(x_1)\psi_a(x_1)\,x_1^2\,dx_1 \cdot$$

$$\underbrace{\int_{-\infty}^{\infty}\psi_b^*(x_2)\,\psi_b(x_2)\,dx_2}_{=1 \text{ (Normalizatua)}} = \int_{-\infty}^{\infty}|\psi_a(x_1)|^2 x_1^2\,dx_1 = \langle x_1^2\rangle_a$$

* $\langle x_2^2\rangle_\psi = \int_{-\infty}^{\infty} |\psi_b(x_2)|^2 x_2^2\, dx_2 = \langle x_2^2\rangle_b$

* $\langle x_1 x_2\rangle_\psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x_1,x_2)|^2 x_1 x_2\, dx_1 dx_2 = \int_{-\infty}^{\infty} |\psi_a(x_1)|^2 x_1\, dx_1 \int_{-\infty}^{\infty} |\psi_b(x_2)|^2 x_2\, dx_2 =$

$\langle x_1\rangle_a \langle x_2\rangle_b$ → ez dago korelaziorik: $\langle x_1 x_2\rangle_\psi = \langle x_1\rangle_a \langle x_2\rangle_b$

↳ Dentsitate probabilitatea banangarria delako
$P(x_1,x_2) = |\psi(x_1,x_2)|^2 = |\psi_a(x_1)|^2 |\psi_b(x_2)|^2$

Errepikatu kalkulu hauek Fermioi eta bosoien kasuan.

Bosoiak ⇒ $\psi_B(x_1,x_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)]$

$\langle (x_1-x_2)^2\rangle_{\psi_B} = \langle x_1^2\rangle_{\psi_B} + \langle x_2^2\rangle_{\psi_B} - 2\langle x_1 x_2\rangle_{\psi_B}$

* $|\psi_B|^2 = \psi_B^* \psi_B = \frac{1}{2}(\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2))(\psi_a^*(x_1)\psi_b^*(x_2) + \psi_b^*(x_1)\psi_a^*(x_2)) =$

$\frac{1}{2}\Big[|\psi_a(x_1)|^2|\psi_b(x_2)|^2 + \psi_a(x_1)\psi_b^*(x_1)\psi_b(x_2)\psi_a^*(x_2) + |\psi_b(x_2)|^2|\psi_a(x_1)|^2 +$

$\psi_b(x_1)\psi_a^*(x_1)\psi_b^*(x_2)\psi_a(x_2)\Big] = \frac{1}{2}\Big[|\psi_a(x_1)|^2|\psi_b(x_2)|^2 + |\psi_b(x_1)|^2|\psi_a(x_2)|^2 +$

$2\,\mathrm{Re}\,[\psi_a(x_1)\psi_b^*(x_1)\psi_b(x_2)\psi_a^*(x_2)]\Big]$

* $\langle x_1^2\rangle_{\psi_B} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 |\psi_B|^2 dx_1\, dx_2 = \frac{1}{2}\int_{-\infty}^{\infty} x_1^2|\psi_a(x_1)|^2 dx_1 + \frac{1}{2}\int_{-\infty}^{\infty} x_1^2|\psi_b(x_1)|^2 dx_1 +$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2\, \mathrm{Re}\,[\cancel{\psi_a(x_1)\psi_b^*(x_1)\psi_b(x_2)\psi_a^*(x_2)}^{\,0}]\, dx_1 dx_2 = \frac{1}{2}[\langle x_1^2\rangle_a + \langle x_1^2\rangle_b]$

↓ ortogonalak $\psi_a$ eta $\psi_b$ *'

* $\langle x_2^2\rangle_{\psi_B} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2^2|\psi_B|^2 dx_1 dx_2 = \frac{1}{2}\int_{-\infty}^{\infty} x_2^2\left[|\psi_b(x_2)|^2 + |\psi_a(x_2)|^2\right] dx_2 = \frac{\langle x_2^2\rangle_a + \langle x_2^2\rangle_b}{2}$

* $\langle x_1 x_2\rangle_{\psi_B} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 |\psi_B|^2 dx_1 dx_2 \neq \langle x_1\rangle\langle x_2\rangle$ korelazioa dago.
(ez dago elkarrekintzarik haien artean baina korrelazionatuta daude dena den)

*' $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2\, \psi_a(x_1)\psi_b^*(x_1)\psi_b(x_2)\psi_a^*(x_2)\, dx_1 dx_2 = \underbrace{\int_{-\infty}^{\infty}\psi_b(x_2)\psi_a^*(x_2)dx_2}_{=0} \int_{-\infty}^{\infty} x_1^2\psi_a(x_1)\psi_b^*(x_1)dx_1 = 0$

$$* \langle x_1 x_2 \rangle_{\psi_B} = \frac{1}{2}\left[\int_{-\infty}^{\infty} x_1 |\psi_a(x_1)|^2 dx_1 \int_{-\infty}^{\infty} x_2 |\psi_b(x_2)|^2 dx_2 + \int_{-\infty}^{\infty} x_1 |\psi_b(x_1)|^2 dx_1 \int_{-\infty}^{\infty} x_2 |\psi_a(x_2)|^2 dx_2\right]$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Re\,(\psi_a(x_1)\psi_b^*(x_1)\psi_b(x_2)\psi_a^*(x_2))\, x_1 x_2\, dx_1 dx_2 = \frac{1}{2}\left[\langle x_1\rangle_a \langle x_2\rangle_b + \langle x_1\rangle_b \langle x_2\rangle_a\right] +$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Re\left[\psi_a(x_1)\psi_b^*(x_1)\psi_b(x_2)\psi_a^*(x_2)\right] x_1 x_2\, dx_1 dx_2$$

Gai gurutzatuak ez dira anulatzen

Ez dago interpretazio klasikorik → gai gurutzatuak daude → simetria edo antisimetriaren ondorioz eskakizunetik.

Fermioiekin gauza bera, baina $Re$ beharrean $-Re$ jarriz ($\langle x_1^2\rangle$ eta $\langle x_2\rangle$ berdinak) ⇒ gai gurutzatuak baina ez dira aldatzen

Orduan, fermioietan batez bestekoa distantzia handiagoa izango da.

17-03-29

- Demagun bi partikula hauen artean (karga, masa.. bera) elkarrekintza bat dugula, Coulombiarra adibidez. Zeinek izango dute energiarik altuena, fotoiek edo bosoiek? $V = \frac{Z^2 e^2}{4\pi\varepsilon_0 |x_1 - x_2|}$ (Hau da, perturbazio txiki bat izango du jatorrizko hamiltondarrean)

Bosoien batez-bestekoa distantzia txikiagoa denez elkarrekintza handiagoa izango da eta beraz energia handiagoa izango dute.

- Demagun 11 elektroi dituzula bi dimentsioko a aldeko potentzial osin infinituan. Zein da zinematiko energia eta endakapena?

Fermioiak dira, bereizteznak. Energiak: $E_{n_1,n_2} = \frac{\hbar^2 \pi^2}{2ma^2}(n_1^2 + n_2^2)$

(Partikula bakoitzarena) eta $|n_1, n_2\rangle = \frac{2}{a} \sin \frac{n_1 \pi x}{a} \sin \frac{n_2 \pi y}{a}$. (Spink gabe)

Oinarrizko egoeran energia minimoa da.

Energia bakoitzaren mailak:

$n_1=1, n_2=3$ ———— , $n_2=1, n_1=3$ ————

$n_1=2, n_2=2$ —↑↓—

$n_1=1, n_2=2$ —↑↓— , $n_1=2, n_2=1$ —↑↓—

$n_1=n_2=1$ —↑↓— Berdin digu haien ordena

Oinarrizko egoeran maila hauetan berdinak dira.

$$E = \frac{\hbar^2 \pi^2}{2ma^2} \left(1+1+1+1+1+4+4+1+1+4+4+1+2(4+4)+2(1+9)+(9+1)\right) = \frac{\hbar^2 \pi^2 70}{2ma^2}$$

$g = 4 \to$ azkenengo mailan 4 aukera:

$n_1=1, n_2=3$ —↑↓— —↑— $n_2=1, n_1=3$ / $n_1=1, n_2=3$ —↑↓— —↓— $n_2=1, n_1=3$

$n_1=1, n_2=3$ —↑— $n_1=3, n_2=1$ —↑↓— / $n_1=1, n_2=3$ —↓— $n_1=3, n_2=1$ —↑↓—

$$\Psi = \sqrt{\frac{2}{a}} \sin \frac{\pi x_1}{a} \sqrt{\frac{2}{a}} \sin \frac{\pi y_1}{a} \chi^+ \sqrt{\frac{2}{a}} \sin \frac{\pi x_2}{a} \sqrt{\frac{2}{a}} \sin \frac{\pi y_2}{a} \chi^- \dots$$ Aukera guztiak,

permutazio guztiak kontuan hartu eta antisimetrizatu.

17-03-30

- Demagun bi elektroi ditugula eta haien arteko elkarrekintzari dagokion hamiltondarra hauxe dela: $\hat{H} = -\alpha \vec{S}_1 \cdot \vec{S}_2$. Zein da oinarrizko egoera?

$$\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2 \Rightarrow \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left[S^2 - S_1^2 - S_2^2\right]$$

Autobektoreak $\Rightarrow$ $(S_1 = S_2 = 1/2)$ $|S_1, S_2, S, m_s\rangle = |S, m_s\rangle$

$$E = -\frac{\alpha}{2}\left[\hbar^2 s(s+1) - \hbar^2 s_1(s_1+1) - \hbar^2 s_2(s_2+1)\right]$$

$s_1 = s_2 = 1/2$ direnez $\Rightarrow E(s) = -\frac{\alpha}{2}\hbar^2\left(s(s+1) - \frac{3}{2}\right)$ , $s \in (|s_1 - s_2|, s_1 + s_2)$

* Energia minimoa $\Leftrightarrow$ oinarrizko egoera $\Leftrightarrow$ $S$ maximoa $\Rightarrow S = s_1 + s_2 = 1$

$$E_0 = -\frac{\alpha}{2}\hbar^2\left(2 - \frac{3}{2}\right) = -\frac{\alpha}{4}\hbar^2 \Rightarrow \text{endekatua} \quad (g=3)$$

* Oinarri bat elektroien autofuntzioak konbinatuz: $\{|\pm\rangle\} \otimes \{|\pm\rangle\} = \{|+,+\rangle,$ $|+,-\rangle, |-,+\rangle, |-,-\rangle\}$ Aurrekoaz konturatuko ez bagina $H$ oinarri horretan garatu beharko genuke eta diagonalizatu: $\vec{S}_1 \cdot \vec{S}_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}$.

$$Ad \Rightarrow \hat{S}_{1z}\hat{S}_{2z} = \hat{S}_{1z} \otimes \hat{S}_{2z} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \overset{*'}{=} \frac{\hbar^2}{4}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$*' \quad A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B\end{pmatrix}$$

* Oinarrizko egoeran $s=1$ da baina $m_s = -1, 0, 1 \Rightarrow g = 3$ endekapena.

- $|1,1\rangle = |+,+\rangle$ (aukera bakarra $m_{s_1} = m_{s_2} = 1/2$ izatea da)
- $|1,-1\rangle = |-,-\rangle$ (aukera bakarra $m_{s_1} = m_{s_2} = -1/2$ izatea da)
- $|1,0\rangle = \frac{1}{\sqrt{2}\hbar} S_-|1,1\rangle = \frac{1}{\sqrt{2}\hbar}(S_{1-} + S_{2-})|+,+\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$

tripletea $\rightarrow$ hauen edozein konbinazio lineala egongo da.

Guztira egoera antisimetrikoa izan behar denez (fermioiak dira) egoera espaziala aukeratzerakoan hau izan behar dugu kontuan. 3 spin egoera hauek (tripletea) simetrikoak direnez ekuazioen espaziala antisimetrikoa izan beharko da.

17-04-03

- Bi elektroi ditugu hasieran singlete egoeran eta $\hat{H} = \omega(\hat{S}_{1x} + \hat{S}_{2x}) = \omega \hat{S}_x$ da.

Zein da $\langle S_{1y} S_{2y} \rangle (t)$?

$|\psi(0)\rangle = |\text{Singlet}\rangle = \frac{1}{\sqrt{2}} (|+,-\rangle - |-,+\rangle) \Rightarrow$ Hau da $s = 0$

Hainbat modu ariketa hau egiteko $\Rightarrow$ H kalkulatu eta diagonalizatu autobalioak eta autobektoreak kalkulatzeko, garatu $|\pm,\pm\rangle = |+\rangle \otimes |-\rangle$ oinarrian $\{|\pm,\pm\rangle_x\}$ oinarrian...

$\longrightarrow$ singletea

- Baina kontutan bagara $s = 0$ denez $S^2$, $\vec{S} = 0$ izango dira berez edozein proiekzio, $S_u$ $(S_x, S_y, S_z, \ldots)$ nulua izango da ere berez

$\hat{H}|\psi(0)\rangle = 0$ izango dugu $\longleftrightarrow$ $|\psi(0)\rangle$ $\hat{H}$-ren autobektorea da.

$$\hat{H}\left(\frac{1}{\sqrt{2}}[|+,-\rangle - |-,+\rangle]\right) = \omega \hat{S}_x \left(\frac{1}{\sqrt{2}}[|+,-\rangle - |-,+\rangle]\right) = 0$$

Orduan, egoera hori $E = 0$ energia espektuko zaio (autobalioa) eta denboraren gorapenean konstante mantenduko da $\rightarrow$ $e^{-Eit/\hbar} = 1$

$$|\psi(t)\rangle = |\psi(0)\rangle = \frac{1}{\sqrt{2}} (|+,-\rangle - |-,+\rangle)$$

- Matrizea garatu nahi izango bagenu $\Rightarrow$ $\hat{S}_x = \hat{S}_{1x} + \hat{S}_{2x}$

$*$ $\hat{S}_{1x} = \hat{S}_{1x} \otimes \mathbb{1} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow \{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\}$ oinarrian

$\hookrightarrow$ Partikula bakoitzaren espazioan

$*$ $\hat{S}_{2x} = \mathbb{1} \otimes \hat{S}_{2x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Orduan $\Rightarrow \hat{H} = \omega \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ 4 x 4 matrizea → ezo kapitulua diagonalizatzea.

$$|\hat{H} - \lambda \mathbb{1}| = 0 \rightarrow \tilde{\lambda} = \frac{2\lambda}{\hbar\omega} \rightarrow |\hat{H} - \tilde{\lambda}\mathbb{1}| = \begin{vmatrix} -\tilde{\lambda} & 1 & 1 & 0 \\ 1 & -\tilde{\lambda} & 0 & 1 \\ 1 & 0 & -\tilde{\lambda} & 1 \\ 0 & 1 & 1 & -\tilde{\lambda} \end{vmatrix} = \begin{vmatrix} -\tilde{\lambda} & 1 & 1 & 0 \\ 0 & -\tilde{\lambda} & 0 & 1 \\ 0 & 0 & -\tilde{\lambda} & 1 \\ \tilde{\lambda} & 1 & 1 & -\tilde{\lambda} \end{vmatrix} =$$

$$-\tilde{\lambda} \begin{vmatrix} -\tilde{\lambda} & 0 & 1 \\ 0 & -\tilde{\lambda} & 1 \\ 1 & 1 & -\tilde{\lambda} \end{vmatrix} + \tilde{\lambda}(-1)^5 \begin{vmatrix} 1 & 1 & 0 \\ -\tilde{\lambda} & 0 & 1 \\ 0 & -\tilde{\lambda} & 1 \end{vmatrix} = -\tilde{\lambda}\left(-\tilde{\lambda}^3 + 2\tilde{\lambda} + 2\tilde{\lambda}\right) = -\tilde{\lambda}\left(-\tilde{\lambda}^3 + 4\tilde{\lambda}\right) =$$

$$-\tilde{\lambda}^2\left(-\tilde{\lambda}^2 + 4\right) = 0 \rightarrow \tilde{\lambda} = 0, \ \tilde{\lambda} = \pm 2 \Rightarrow \lambda = \begin{cases} 0 & g = 2 \\ \hbar\omega \\ -\hbar\omega \end{cases}$$

(espero genuen bezala $S_x = \hbar, 0, -\hbar$ )

- $\tilde{\lambda} = 0 \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b+c \\ a+d \\ a+d \\ b+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow b = -c, \ a = -d$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|+,+\rangle - |-,-\rangle\right)$$

↳ singletea

- $\tilde{\lambda} = 2 \rightarrow \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -2a+b+c \\ a-2b+d \\ a-2c+d \\ b+c-2d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow$ $b+c = 2a$, $d = 2b - a$, $d = 2c - a$ $\rightarrow b = c$ $\rightarrow a = c = d$

$$|\psi_3\rangle = \frac{1}{2}\left(|+,+\rangle + |+,-\rangle + |-,+\rangle + |-,-\rangle\right)$$

- $\tilde{\lambda} = -2 \rightarrow \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2a+b+c \\ a+2b+d \\ a+2c+d \\ b+c+2d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow$ $b+c = -2a$, $d = -(a+2b) = -(a+2c) \rightarrow$ $b = c \rightarrow a = -b$, $d = -(a-2a) = a$

$$|\psi_4\rangle = \frac{1}{2}\left(|+,+\rangle - |+,-\rangle - |-,+\rangle + |-,-\rangle\right)$$

Bestela $\Rightarrow |\pm\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle_x \pm |-\rangle_x\right)$ direz gure egoera (singletea) $S_x$-ren autofuntzioaren garatuko dugu honela:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right) = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x) \otimes \frac{1}{\sqrt{2}}(|+\rangle_x - |-\rangle_x) + \right.$$

$$\left. - \frac{1}{\sqrt{2}}(|+\rangle_x - |-\rangle_x) \otimes \frac{1}{\sqrt{2}}(|+\rangle_x + |-\rangle_x)\right] = \dots = \frac{1}{\sqrt{2}}\left(|+\rangle_x \otimes |-\rangle_x - |-\rangle_x \otimes |+\rangle_x\right)$$

"Singletea X-n"

Hori $\hat{S}_x = \hat{S}_{1x} + \hat{S}_{2x}$ aplikatzean 0 lortzen dugu → autobektorea da

- Orduan orain $\langle S_{1y} S_{2y}\rangle(t)$ kalkulatzeko lehenengo $\hat{S}_{1y}\hat{S}_{2y}$ matrizea kalkulatuko dugu.

$$\hat{S}_{1y}\hat{S}_{2y} = \hat{S}_{1y} \otimes \hat{S}_{2y} = \frac{\hbar i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \frac{\hbar i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =$$

$$\frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \boxed{\langle \hat{S}_{1y}\hat{S}_{2y}\rangle(t)} = \left(0 \ \frac{1}{\sqrt{2}} \ -\frac{1}{\sqrt{2}} \ 0\right)(\hat{S}_{1y}\hat{S}_{2y})\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} =$$

$$\frac{\hbar^2}{4}\left(0 \ -1/\sqrt{2} \ 1/\sqrt{2} \ 0\right)\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}\left(-\frac{1}{2} - \frac{1}{2}\right) = \boxed{-\frac{\hbar^2}{4}}$$

17-04-07

- Bi partikula ditugu $j_1 = 1$ eta $j_2 = 1/2$ izanik. $t=0$ aldiunean $J_{1z} = 0$ eta $J_{2z} = -\hbar/2$ dira. $H = A\vec{J}_1 \cdot \vec{J}_2$ izanik zein da $\langle \vec{J}_1\rangle(t)$?

$m_1 \Rightarrow -1, 0, 1$ , $m_2 \Rightarrow -\frac{1}{2}, 1/2$ → $t=0$ aldiunean $m_1 = 0$ eta $m_2 = -1/2$

$H = A\vec{J}_1 \cdot \vec{J}_2 = \frac{A}{2}[J^2 - J_1^2 - J_2^2] \Rightarrow$ autobektoreak $|j, m\rangle$

$j = 1/2, 3/2$ → garatu hasierako egoera $|j_1 = 1, m_1 = 0; j_2 = 1/2, m_2 = -1/2\rangle$

$\{|j, m\rangle\}$ oinarrian.

$m = m_1 + m_2 = -1/2$

$|j_1 = 1, \ m_1 = 0 \ ; \ j_2 = 1/2, \ m_2 = -1/2 \rangle = \alpha | 1/2, -1/2\rangle + \beta | 3/2, -1/2\rangle$

- $|3/2, -3/2\rangle = |j_1 = 1, \ m_1 = -1, \ j_2 = 1/2, \ m_2 = -1/2\rangle$

- $|3/2, -1/2\rangle = \dfrac{J_+}{\hbar\sqrt{3}} |3/2, -3/2\rangle = \dfrac{1}{\hbar\sqrt{3}} (J_{1+} + J_{2+}) |m_1 = -1, \ m_2 = -1/2\rangle =$

$\dfrac{1}{\cancel{\hbar}\sqrt{3}} \left( \cancel{\hbar}\sqrt{2} \, |m_1 = 0, \ m_2 = -1/2\rangle + \cancel{\hbar} \, |m_1 = -1, \ m_2 = 1/2\rangle \right)$

- $|1/2, -1/2\rangle = a \ |m_1 = 0, \ m_2 = -1/2\rangle + b \ |m_1 = -1, \ m_2 = 1/2\rangle$

$\langle 1/2, -1/2 | 3/2, -1/2\rangle = \sqrt{\frac{2}{3}} \, a + \dfrac{b}{\sqrt{3}} = 0 \rightarrow \quad a = -\sqrt{2}\, b \ \rightarrow$

$|1/2, -1/2\rangle = \dfrac{1}{\sqrt{3}} \ |m_1 = 0, \ m_2 = -1/2\rangle - \sqrt{\dfrac{2}{3}} \ |m_1 = -1, \ m_2 = 1/2\rangle$

Beraz $\Rightarrow$ $|j_1 = 1, \ m_1 = 0; \ j_2 = 1/2, \ m_2 = -1/2\rangle = \dfrac{1}{\sqrt{3}} \, |1/2, -1/2\rangle + \dfrac{\sqrt{2}}{\sqrt{3}} \, |3/2, -1/2\rangle$

$\dfrac{\beta}{\cancel{\sqrt{3}}} - \alpha \sqrt{\dfrac{2}{\cancel{3}}} = 0 \rightarrow \beta = \sqrt{2}\alpha \ , \qquad \alpha^2 + \beta^2 = 2\alpha^2 + \alpha^2 = 3\alpha^2 = 1 \rightarrow \alpha = \dfrac{1}{\sqrt{3}}$

* H-ren auto balioak $\rightarrow$ $E_j = \dfrac{A}{2} \left[ \hbar^2 j(j+1) - 2\hbar^2 - \dfrac{3}{4} \hbar^2 \right] = \dfrac{A\hbar^2}{2} \left[ j(j+1) - \dfrac{11}{4} \right]$

Beraz $\Rightarrow$ $|\Psi(0)\rangle = |m_1 = 0, \ m_2 = -1/2\rangle \xrightarrow{t} |\Psi(t)\rangle = \dfrac{1}{\sqrt{3}} \, |1/2, -1/2\rangle e^{+A\hbar i t} +$

$\sqrt{\dfrac{2}{3}} \ |3/2, -1/2\rangle \ e^{-A\hbar i t/2}$

Orduan $\Rightarrow$ $\langle \vec{J}_1 \rangle_{(t)} = \langle \Psi(t) | \vec{J}_1 | \Psi(t)\rangle = \langle \Psi(t) | J_{1x} | \Psi(t)\rangle \, \hat{i} \ +$

$$\langle \psi(t) | J_{1y} | \psi(t) \rangle \hat{j} + \langle \psi(t) | J_{1z} | \psi(t) \rangle \hat{k}$$

- $|\psi(t)\rangle = \frac{1}{\sqrt{3}} |1/2, -1/2\rangle e^{A\hbar i t} + \sqrt{\frac{2}{3}} |3/2, -1/2\rangle e^{-iA\hbar t/2} =$

$$e^{A\hbar i t}\left[\frac{1}{3}|m_1=0, m_2=-1/2\rangle - \frac{\sqrt{2}}{3}|m_1=-1, m_2=1/2\rangle\right] + e^{-iA\hbar t/2}\left[\frac{2}{3}|m_1=0, m_2=-1/2\rangle + \frac{\sqrt{2}}{3}|m_1=-1, m_2=1/2\rangle\right] =$$

$$|m_1=0, m_2=-1/2\rangle \frac{1}{3}\underbrace{\left(e^{iA\hbar t} + 2e^{-iA\hbar t/2}\right)}_{a} + |m_1=-1, m_2=1/2\rangle \frac{\sqrt{2}}{3}\underbrace{\left(e^{-iA\hbar t/2} - e^{iA\hbar t}\right)}_{b}$$

$\hat{J}_{1x}$ , $\hat{J}_{1y}$ eta $\hat{J}_{1z}$ kalkulatu ditugu.

$\{|1,1/2\rangle, |1,-1/2\rangle, |0,1/2\rangle, |0,-1/2\rangle, |-1,1/2\rangle, |-1,-1/2\rangle\}$

* $$\hat{J}_{1x} = \hat{J}_{1x} \otimes \mathbb{1} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&1&0 \\ 0&1&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}$$

* $$\hat{J}_{1y} = \hat{J}_{1y} \otimes \mathbb{1} = \frac{\hbar}{\sqrt{2}} i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar i}{\sqrt{2}}\begin{pmatrix} 0&0&-1&0&0&0 \\ 0&0&0&-1&0&0 \\ 1&0&0&0&-1&0 \\ 0&1&0&0&0&-1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}$$

* $$\hat{J}_{1z} = \hat{J}_{1z} \otimes \mathbb{1} = \hbar\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hbar\begin{pmatrix} -1&0&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$$

- $$\langle \psi(t) | J_{1x} | \psi(t) \rangle = (0\ 0\ 0\ a^*\ b^*\ 0)\, J_{1x} \begin{pmatrix} 0 \\ 0 \\ 0 \\ a^* \\ b^* \\ 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}(0\ a^*\ b^*\ 0\ 0\ a^*)\begin{pmatrix} 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \end{pmatrix} =$$

$$\frac{\hbar}{\sqrt{2}} \cdot 0 = 0$$

- $$\langle \psi(t) | J_{1y} | \psi(t) \rangle = (0\ 0\ 0\ a^*\ b^*\ 0)\, J_{1y} \begin{pmatrix} 0 \\ 0 \\ 0 \\ a^* \\ b^* \\ 0 \end{pmatrix} = \frac{\hbar i}{\sqrt{2}}(0\ a^*\ b^*\ 0\ -a^*)\begin{pmatrix} 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \end{pmatrix} = 0$$

• $\langle\Psi(t)|J_{1t}|\Psi(t)\rangle = (0\ 0\ 0\ a^*\ b^*\ 0)\ J_{1t}\begin{pmatrix}0\\0\\0\\a^*\\b^*\\0\end{pmatrix} = \hbar(0\ 0\ 0\ 0\ b^*\ 0)\begin{pmatrix}0\\0\\0\\a\\b\\0\end{pmatrix} =$

$\hbar|b|^2 = \hbar\frac{2}{9}\left|\left(e^{-iA\hbar t/2} - e^{iA\hbar t}\right)\right|^2 =$

$\frac{2\hbar}{9}\left(e^{-iA\hbar t/2} - e^{iA\hbar t}\right)\left(e^{iA\hbar t/2} - e^{-iA\hbar t}\right) = \frac{2\hbar}{9}\Big(1+1+$

$-e^{-3iA\hbar t/2} - e^{3iA\hbar t/2}\Big) = \frac{2\hbar}{9}\left(2-2\cos\left(\frac{3A\hbar t}{2}\right)\right) =$

$\frac{4\hbar}{9}\left(1-\cos\left(\frac{3A\hbar t}{2}\right)\right)$

17-04-10

• Helio atomoa dugu ⇒ 1 é 1s egoeran eta bestea $n, l$ egoeran.

kalkulatu honako elementua orbitalez: $\langle\Psi_0|\vec{r}_1+\vec{r}_2|\Psi_p\rangle$

non $\Psi_0$ orto egoera den eta $\Psi_p$ parahelio egoera.

$* \Psi_p = \frac{1}{\sqrt{2}}\left(|100\rangle_1|n\ l\ m_l\rangle_2 - |n\ l\ m_l\rangle_1|1\ 0\ 0\rangle_2\right)\otimes|\chi_{0,0}\rangle$

$* \Psi_0 = \frac{1}{\sqrt{2}}\left(|100\rangle_1|n\ l\ m_l\rangle_2 + |n\ l\ m_l\rangle_1|1\ 0\ 0\rangle_2\right)\otimes|\chi_{1,m_s}\rangle$

$\vec{r}_1+\vec{r}_2 - K$ ez du eragiten spin egoeran eta spin egoerak ortogonalak direnez biderkadura edozein 0 izango da

$\langle\Psi_0|\vec{r}_1+\vec{r}_2|\Psi_p\rangle = 0$

espaziala $\langle\Psi_0^S|\vec{r}_1+\vec{r}_2|\Psi_p^S\rangle\cdot\underbrace{\langle\chi_{0,0}|\chi_{1,m_s}\rangle}_{0}$

• Aurreko ariketan spina kontuan izango ez bagenu, zein izango litzateke emaitza?

$\langle\Psi_0^S|\vec{r}_1+\vec{r}_2|\Psi_p^S\rangle$?

$\begin{cases}\vec{r}_1\Rightarrow\vec{r}_2\\ \vec{r}_2\Rightarrow\vec{r}_1\end{cases}$ isinez aurrekoaren berdina

$\langle\Psi_0^S|\vec{r}_1+\vec{r}_2|\Psi_p^S\rangle = \frac{1}{2}\Big[\int d\vec{r}_1 d\vec{r}_2\,|\Psi_{100}(1)|^2|\Psi_{nlm_l}(2)|^2(\vec{r}_1+\vec{r}_2) - \int d\vec{r}_1 d\vec{r}_2|\Psi_{nlm}(1)|^2|\Psi_{100}(2)|^2(\vec{r}_1+\vec{r}_2)$

$\int d\vec{r}_1 d\vec{r}_2\,\Psi_{100}^*(1)\Psi_{nlm_l}^*(2)\Psi_{nlm}(1)\Psi_{100}(2)(\vec{r}_1+\vec{r}_2) - \int d\vec{r}_1 d\vec{r}_2\,\Psi_{nlml}^*(1)\Psi_{100}^*(2)\Psi_{100}(1)\Psi_{nlm}(2)(\vec{r}_1+\vec{r}_2) =$

$0 \Rightarrow$ aldagai aldaketa bat eginez $\vec{r}_1 \leftrightarrow \vec{r}_2$ lehenengo bi integralak

berdinak dira eta anulatuko dira eta beste biak baita.

Adibidez $\Rightarrow$ $\vec{p} = -e\vec{r}_1 - e\vec{r}_2 = -e(\vec{r}_1 + \vec{r}_2)$ (Momentu dipolar elektrikoa)

Eremu elektriko bat aplikatzen badugu, hori dagokion elkarrekintza hauxe da:

$$H_{elk.} = -\vec{p} \cdot \vec{E} = e(\vec{r}_1 + \vec{r}_2) \cdot \vec{E}$$

Eremu horretan $\psi_p$-tik $\psi_0$-ra trantsizioa egiteko probabilitatea:

$$P \propto |\langle \psi_0 | H_{elk} | \psi_p \rangle|^2$$

Beraz, aurreko emaitzak kontuan hartuz probabilitatea nulua izango da.

- Demagun bi elektroi ditugula eta hauxe dela gure hamiltondarra

perturbaziorik gabe: $H_0 = -\alpha (\hat{S}_{1z} + \hat{S}_{2z}) \Rightarrow \{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\}$

da $H_0$-ren autobektoreen oinarria.

$H_{elk} = \beta (\hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y})$ elkarrekintza gehitzen badugu (perturbazioa)

zenbat aldatzen diren $H_0$-ren autobalioak perturbazio teoriaren lehenengo

hurbilketan.

$E_i^0 = -\alpha\hbar(m_{s_1} + m_{s_2}) = -\alpha\hbar m_s$ ($H_0$-ren autobalioak) $m = \begin{cases} 1 \\ 0 \quad g=2 \\ -1 \end{cases}$

- $\hat{S}_{1x}\hat{S}_{2x} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

- $\hat{S}_{1y}\hat{S}_{2y} = \frac{\hbar i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \frac{\hbar i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

$$H_{elk} = \beta \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $E_1^0 = -\alpha\hbar \Rightarrow m=1$ denean $\Rightarrow |\psi_1^0\rangle = |+,+\rangle$

$E_1 = E_1^0 + \langle \psi_1^0 | H_{elk} | \psi_1^0 \rangle \overset{*}{=} E_1^0 = -\alpha\hbar$ (ez dago zuzenketarik)

$* \; \langle \psi_1^0 | H_{elk} | \psi_1^0 \rangle = (1\ 0\ 0\ 0) \left( \beta \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} =$

$-\alpha \frac{\hbar^2}{4} (0\ 0\ 0\ 0) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$

- $E_2^0 = +\alpha\hbar \Rightarrow m=-1$ denean $\Rightarrow |\psi_2^0\rangle = |-,-\rangle$

$E_2 = E_2^0 + \langle \psi_2^0 | H_{elk} | \psi_2^0 \rangle \overset{*}{=} E_2^0 = \alpha\hbar$

$* \; \langle \psi_2^0 | H_{elk} | \psi_2^0 \rangle = (0\ 0\ 0\ 1) \left( \beta \frac{\hbar^2}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$

- $m=0$ denean $\Rightarrow$ endakapena $H_{elk}^{(3)}$ diagonalizatu $\Rightarrow$

$H_{elk}^{(3)} = \beta \frac{\hbar^2}{4} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \Rightarrow |H_{elk} - \mathbb{1}\lambda| = \begin{vmatrix} -\lambda & 2\beta\hbar^2/4 \\ \frac{2\beta\hbar^2}{4} & -\lambda \end{vmatrix} = \lambda^2 - \frac{\beta^2\hbar^2}{4} = 0 \rightarrow$

$\lambda = \pm \frac{\beta\hbar^2}{2}$

$\hookrightarrow E_3 = E_3^0 + \frac{\beta\hbar^2}{2} = \frac{\beta\hbar^2}{2}$ , $E_4 = E_3^0 - \frac{\beta\hbar^2}{2} = -\frac{\beta\hbar^2}{2}$

17-04-24

- $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^{10}\ 4s^2\ 4p^1\ 4d^1$

Oinarrizko egoeran dago atomoa? Ez badago zein da oinarrizko egoera?

Zein atomo da?

$Z = 32 \Rightarrow$ Germanioa

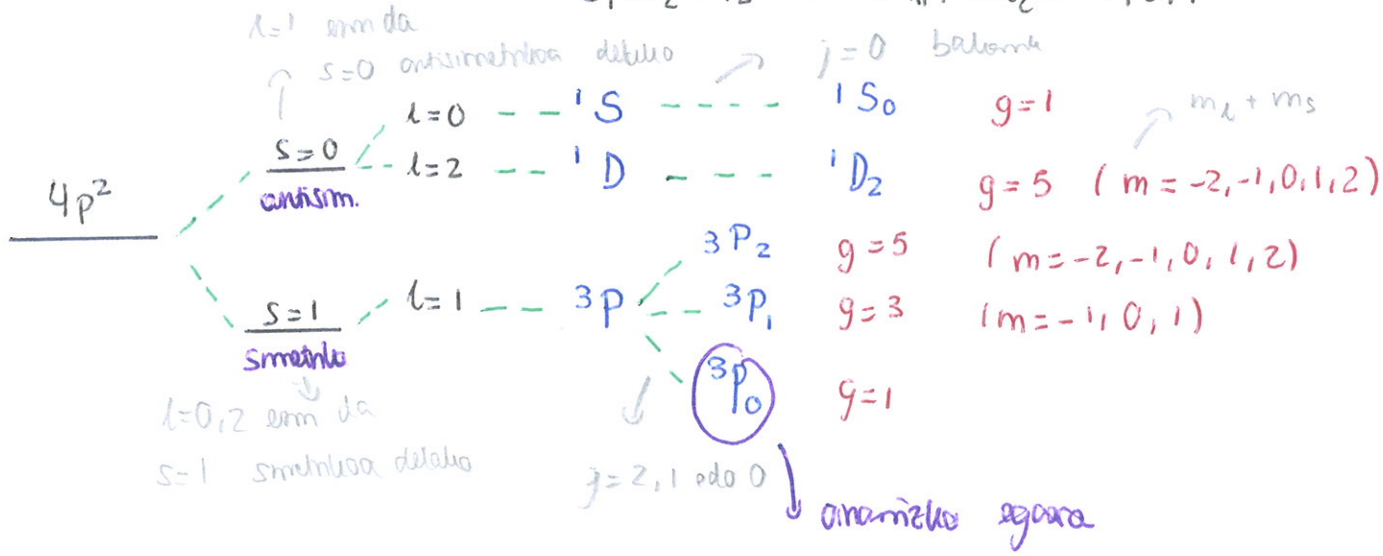
Ez dago oinarrizko egoeran, azkenengo $e^-$-a $4d$-n dagoelako.

Oinarrizko egoera $\Rightarrow 1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^{10}\ 4s^2\ 4p^2$

Endekapena dago, konfigurazio honekin egoera asko daudelako.

Russel-Saunders-en akoplamendua kontuan hartuz energia-mailak banatuko ditugu. $4p^2 \Rightarrow n_1 = n_2 = 4,\ l_1 = l_2 = 1 \quad ms_1, ms_2 = \pm 1/2$

$s_1 = s_2 = 1/2, \quad ml_1, ml_2 = -1, 0, 1$




Demagun $s=1$ denean $l=2$ posible izango litzatekeela $\Rightarrow$ orduan $^3D_3$ izango genuke. Halaber, demagun oinarria atomo hidrogenoideoen autofuntzioek osatzen dutela: $R_{41} Y_1^{m_l} \chi_{1/2, m_s}$ ($n=4$ eta $l=1$ dugu)

Halaber, simplifikatzeko demagun $m = m_s + m_l = 3$ maximoa dela.

Horrela 2 elektroi ditugu → oinarria $\{ R_{41}(1) R_{41}(2) Y_1^{m_{l_1}}(1) \chi_{1/2, m_{s_1}}^{(1)} Y_1^{m_{l_2}}(2) \chi_{1/2, m_{s_2}}^{(2)} \}$.

$s_2 = s_1 = 1/2$ da eta $l_1 = l_2 = 1$ dugu → $S = 0, 1$ eta $L = 2, 1, 0$ → $j = 3, 2, 1, 0$

$m = m_s + m_l \rightarrow m_{max} = m_{s\,max} + m_{l\,max}$ $\begin{cases} m_{s\,max} = 1 \\ m_{l\,max} = 2 \end{cases}$

$m_{s\,max} = 1 \Rightarrow$ Spina ⇒ $|++\rangle$ (simetrikoa)

$m_{l\,max} = 2 \Rightarrow$ Espaziala ⇒ $Y_1^1(1)\, Y_1^1(2)$ (simetrikoa)

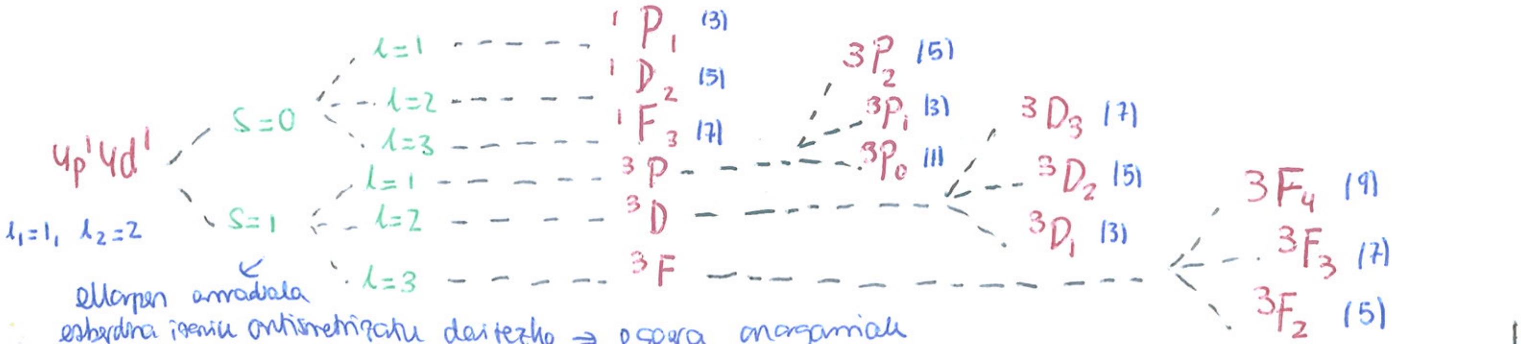
Beraz ⇒ $\Psi = R_{41}(1) R_{41}(2) Y_1^1(1) Y_1^1(2) \chi_{1/2,1}^{(1)} \chi_{1/2,1}^{(2)}$ (simetrikoa)

Ezin da antisimetrizatu baina kasu hipotetikoa da.

$4p^2$ beharrean $4p^1 5p^1$ izango bagenu elkarpen erradialak jdaste ahal izango genuke antisimetrizatzeko (gainontzekoa berdina izango litzateke). ⇒ egoera hori posiblea izango litzateke.

- Demagun Ge dugula egoera kitzikatuan: $1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p^6\, 3d^{10}\, 4s^2\, 4p^1\, 4d^1$

LS akoplamendua kontuan hartuz zeintzuk dira energia-mailak?



17-04-27

- ¿$^1P_1$ egoerak paritate definitua dauka? Hau da, simetrikoa edo antisimetrikoa da?

$j=1,\ s=0,\ l=1 \rightarrow m=-1,0,1$

$4p^1 4d^1 \Rightarrow Y_{l_1=1}^{m_{l_1}} Y_{l_2=2}^{m_{l_2}}$ → hauek konbinazioz antisimetrizatu daitezke

Egoera guztiek badute paritate definitua eta guztiena $-1$ da ⇒ antisimetrikoak

- Demagun Ge atomikoa egoeran dagoela ($^3P_0$)

$H=-\vec{p}\cdot\vec{E}$ perturbazioa gehitzean atomikoa egoeratik alde egin ahal izango du, trantsizioa eginez. Trantsizio hauek ordea, ezin dira edonolakoak izan, adibidez spina ezin da aldatu (H-k ez duelako zerikusirik honekin; atomikoa mantendu behar da). Arauak:

∘ $\Delta S=0$ ∘ $\Delta L=0,\pm1$ ∘ $\Delta J=0,\pm1$ ($0\to0$ ezinezkoa)

Zeintzuk dira trantsizio posibleak egoera kitzikatua goikoa izanik?

$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^2 \Rightarrow 1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^1 4d^1$

↳ $^3P_0$ $(L=1)$

- $\Delta S=0 \rightarrow S=1$ izan behar da $\langle$ $L=1$, $L=2$, $\cancel{L=3}$
- $\Delta L=0,\pm1$ izan behar denez $L=3$ ez da posiblea izango
- $\Delta J=0,\pm1$ izan behar da eta $0\to0$ posible ez denez $J=1$ izan behar da.

Beraz, egoera posibleak: $^3P_1$ eta $^3D_1$

17-05-05

• Dimentsio bakarreko $H_2^+$ molekula aztertuko dugu. Elektroiak nukleoekin duten elkarrekintza Dirac-en deltatzat hartuko dugu:

$$V(x) = -\alpha\,\delta(x+R/2) - \alpha\,\delta(x-R/2)$$

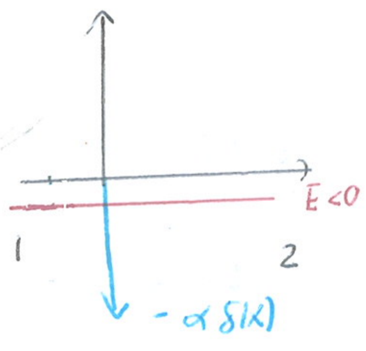
$-R/2$ eta $R/2$ ⇒ bi nukleoen posizioak

Nukleoen arteko elkarrekintza arbuiatuko dugu. LCAO hurbilketa erabiliz oinarrizko energia kalkulatu:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

Lehenengo, dimentsio batean eta $e^-$ eta nukleoaren arteko elkarrekintza $-\alpha\,\delta(x-a)$ izanik orbitalak kalkulatu behar ditugu (ez dira 1s, 2s, ...).

$$* \quad \hat{H}_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\,\delta(x) \Rightarrow \hat{H}_1\psi_1 = E\psi_1$$




• $x<0 \Rightarrow \psi_1 = Ae^{Kx} + \cancel{Be^{-Kx}}$ $\quad K = \frac{\sqrt{-2mE}}{\hbar}$

• $x>0 \Rightarrow \psi_2 = \cancel{Ce^{Kx}} + De^{-Kx}$ $\quad K = \frac{\sqrt{-2mE}}{\hbar}$

$x \to \pm\infty \quad \psi \to 0 \Rightarrow B = 0, \quad C = 0$

Jarraitasuna ⇒ $\psi_1(x=0) = \psi_2(x=0) \rightarrow A = D$

*Normalizatu ⇒ $1 = 2\int_0^\infty D^2 e^{-2Kx}\,dx = \frac{D^2}{K} \rightarrow D = \sqrt{K}$

$\frac{\partial \psi}{\partial x}$ -ren ez - jarraitasuna:

$$\lim_{\varepsilon \to 0} \left\{ \frac{-\hbar^2}{2m} \int_{-\varepsilon}^{\varepsilon} \frac{d^2\psi}{dx^2} dx - \int_{-\varepsilon}^{\varepsilon} \alpha \delta(x) \psi \, dx = \int_{-\varepsilon}^{\varepsilon} E \cdot \psi \, dx \right\} =$$

$$\lim_{\varepsilon \to 0} \left\{ -\frac{\hbar^2}{2m} \left( \frac{d\psi_2}{dx}\Big|_{\varepsilon} - \frac{d\psi_1}{dx}\Big|_{-\varepsilon} \right) - \alpha \psi(0) \simeq E \cdot \psi(0) \cdot 2\varepsilon \right\} \Rightarrow$$

$A = D$

$$0 = -\frac{\hbar^2}{2m} \left( -KD - KA \right) - \alpha A \rightarrow \alpha \cancel{A} = 2K\cancel{A} \frac{\hbar^2}{2m} \Rightarrow K = \frac{m\alpha}{\hbar^2} \Rightarrow$$

$$K = \frac{m\alpha}{\hbar^{\cancel{2}}} = \frac{\sqrt{-2mE}}{\cancel{\hbar}} \Rightarrow \left( \frac{m\alpha}{\hbar} \right)^2 = -2mE \rightarrow E = -\frac{m}{2} \frac{\alpha^2}{\hbar^2}$$

$$\psi = \begin{cases} \sqrt{K} e^{Kx} & x \leq 0 \\ \sqrt{K} e^{-Kx} & x > 0 \end{cases} \qquad K = \frac{m\alpha}{\hbar^2}$$

Egoera lotu bakarra

- Hau $x = -R/2$ eta $x = R/2$ -n zentratuta izango da gure orbitala

LCAO aplikatzeko: $\{ \psi_1, \psi_2 \}$

$$\Psi = c_1 \psi_1 + c_2 \psi_2 = c_1 \sqrt{K} e^{-K|x + R/2|} + c_2 \sqrt{K} e^{-K|x - R/2|}$$

$\hat{H} \Psi = \varepsilon \Psi \rightarrow \varepsilon \equiv$ orbitalaren energia ; $H$ matrizea kalkulatu

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{11} \end{pmatrix} \Rightarrow \text{Hau diagonalizatu}$$

↳ hermitikoa

$$H_{11} = \langle \Psi_1 | \hat{H} | \Psi_1 \rangle = \langle \Psi_1 | \underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x+R/2)}_{\Psi_1 \text{ haren autofuntzioa da}} - \alpha\delta(x-R/2) | \Psi_1 \rangle = -\frac{\alpha^2 m}{2\hbar^2} +$$

$$-\alpha \langle \Psi_1 | \delta(x-R/2) | \Psi_1 \rangle \overset{*^1}{=} -\frac{\alpha^2 m}{2\hbar^2} - \alpha K e^{-2KR} = -\frac{\alpha^2 m}{2\hbar^2} - \frac{m\alpha^2}{\hbar^2} e^{-2KR}$$

$$*^1 \quad \langle \Psi_1 | \delta(x-R/2) | \Psi_1 \rangle = \int_{-\infty}^{\infty} K e^{-2K|x+R/2|} \delta(x-R/2)\, dx = K e^{-2K|R/2+R/2|} =$$

$$K e^{-2KR}$$

$$H_{12} = \langle \Psi_1 | \hat{H} | \Psi_2 \rangle = \langle \Psi_1 | \underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x-R/2)}_{\Psi_2 \text{ haren autofuntzioa da}} - \alpha\delta(x+R/2) | \Psi_2 \rangle =$$

$$-\frac{\alpha^2 m}{2\hbar^2} \langle \Psi_1 | \Psi_2 \rangle - \alpha \langle \Psi_1 | \delta(x+R/2) | \Psi_2 \rangle \overset{*^2}{=} -\frac{\alpha^2 m}{2\hbar^2} e^{-KR}\left(1 + \frac{Rm\alpha}{\hbar^2}\right) +$$

$$-\frac{m\alpha^2}{\hbar^2} e^{-KR} = -\frac{m\alpha^2}{\hbar^2}\left(\left(\frac{1}{2} + \frac{Rm\alpha}{2\hbar^2}\right) e^{-KR} + e^{-KR}\right)$$

$$*^2 \langle \Psi_1 | \delta(x+R/2) | \Psi_2 \rangle = \int_{-\infty}^{\infty} K e^{-K|x+R/2|} e^{-K|x-R/2|} \delta(x+R/2)\, dx = K e^{-KR}$$

$$\langle \Psi_1 | \Psi_2 \rangle = K \int_{-\infty}^{\infty} e^{-K|x+R/2|} e^{-K|x-R/2|}\, dx = K \int_{-\infty}^{-R/2} e^{+K(x+\cancel{R/2})} e^{K(x-\cancel{R/2})}\, dx +$$

$$K \int_{-R/2}^{R/2} e^{-K(x+R/2)} e^{K(x-R/2)}\, dx + K \int_{R/2}^{\infty} e^{-K(x+\cancel{R/2})} e^{-K(x-\cancel{R/2})}\, dx = e^{-KR} KR +$$

$$\frac{\cancel{K} e^{-KR}}{2\cancel{K}} + \frac{e^{-KR}}{2\cancel{K}} \cancel{K} = e^{-KR} + KRe^{-KR} = e^{-KR}(1+KR) = e^{-KR}\left(1+\frac{Rm\alpha}{\hbar^2}\right)$$

$$S = \langle \Psi_1 | \Psi_2 \rangle = e^{-KR}\left(1+\frac{Rm\alpha}{\hbar^2}\right)$$

Bruk $\Rightarrow$ $$H = -\frac{m\alpha^2}{2\hbar^2}\begin{pmatrix} 1+2e^{-2KR} & \left(1+\frac{Rm\alpha}{\hbar^2}\right)e^{-KR}+2e^{-KR} \\ \left(1+\frac{Rm\alpha}{\hbar^2}\right)e^{-KR}+2e^{-KR} & 1+2e^{-2KR} \end{pmatrix} \Rightarrow H\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \varepsilon \tilde{S}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\varepsilon^{\pm} = \frac{H_{11} \pm H_{12}}{1 \pm S}$$

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