2020. ORD. PA

1 A) suppose an intrp. pol: $P_n(x)$ → and its table of DIV. DIFF. of a function $f(x)$ at nodes $\{x_0, x_1, \dots, x_{n-1}, x_n\}$

If a node greater than x_{n-1} but less than x_n is added. Where and how would it be best to add it in order to obtain the table of $P_{n+1}(x)$.

What would be the necessary aditional calculations be

- $x_{n-1} < x_{n+1} < x_n$ → It doesn't affect to the polynomial adding x_{n+1} in that order
- The best way to reuse the previous table and add it at the bottom

B) Define the finite diff. $\Delta^n f(x)$ for any order n and express $\Delta^3 f(x)$ in terms of the function f evaluated at necessary points

$$\Delta^0 f(x) = f(x)$$

$$\Delta^1 f(x) = \Delta f(x) = f(x+h) - f(x)$$

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x+h) - f(x)) = [f(x+2h) - f(x+h)] - [f(x+h) - f(x)] = f(x+2h) - 2f(x+h) + f(x)$$

$$\boxed{\Delta^3 f(x)} = \Delta(\Delta^2 f(x)) = \Delta^2 f(x+h) - \Delta^2 f(x) = [f(x+3h) - 2f(x+2h) + f(x+h)] - [f(x+2h) - 2f(x+h) + f(x)] =$$

$$= \boxed{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}$$

$\vdots$

$$\boxed{\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)}$$

C) Justify why a truncated Taylor series expansion is an osculating polynom'r

When working with osculating polynomials we will have some nodes, with their value of $f(x)$ and for some of the nodes we will also have the values of their derivatives which means that the number of the main diagonal up to a point is going to be the derivative over the number of der. factorial:

example

z_i	$f_{i,0}$	$f_{i,1}$	$f_{i,2}$	f_{i3}	f_{i4}
x_0	f_0	–	–	–	–
x_0	f_0	f_0'	–	–	–
x_0	f_0	f_0'	$f_0''/2!$	–	–
x_1	f_1	$(f_1-f_0)/(x_1-x_0)$	x	x	–
x_1	f_1	f_1'	x	x	x

so if we just take that we only have one node x_0, our polynomial will be of the form: $P_2(x) = f_0 + f_0' \cdot (x-x_0) + \dfrac{f_0''(x-x_0)^2}{2!}$

If we now carry out the expansion of a Taylor polynomial of order 2:

$P_{Taylor(2)} = \dfrac{f(x_0)}{0!}(x-x_0)^0 + \dfrac{f'(x_0)}{1!}(x-x_0)^1 + \dfrac{f''(x_0)}{2!}(x-x_0)^2$ and its exactly as the previous one belonging to the osc. pol.

2 $\quad Shi(x) = \displaystyle\int_0^x \frac{sh(t)}{t}\,dt$

0'1 0'1 3'8288

x	3'5	3'6	3'7	3'8	3'9
Shi(x)	6'9662	7'4562	7'983	8'5498	9'16

Let $x = 3'8288$

A) pol. of degree 3 to optimally aprox x.

$\hookrightarrow$ 4 nodes

B) Error?

* 4 dec. rounding

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2.

A)

x_i	$\Delta_{0,i}$	$\Delta_{1,i}$	$\Delta_{i,2}$	$\Delta_{i,3}$
3'6	7'4562	—	—	—
3'7	7'983	0'5268	—	—
3'8	8'5498	0'5668	0'04	—
3'9	9'16	0'6102	0'0439	0'0034

. finite diff

$x = 3'6 + 0'1\,t$

$\hookrightarrow t = \frac{3'8288 - 3'6}{0'1} = 2'288$

$$q_3(t) = \frac{7'4562}{0!} + \frac{0'5268\,t}{1!} + \frac{0'04\,t(t-1)}{2!} + \frac{0'0034\,t(t-1)(t-2)}{3!}$$

$$q_3(t) = \underbrace{7'4562 + \underbrace{t\left[\underbrace{0'5268 + \underbrace{\frac{(t-1)}{2}\underbrace{\left(0'04 + \underbrace{\frac{0'0034\,(t-2)}{6}}_{0'0002}\right)}_{0'0402}}_{0'0259}}_{0'5527}\right]}_{1'2646}}_{8'7208}$$

$t = 2'288$

$$\boxed{q_3(2'28) = 8'7208}$$

B)

y_i					
3'5	6'9662	0'49	0'0368	0'0032	0'0002

$\Delta_{i,4} = 0'0002$

$$\left[e = \frac{\Delta^{(4)} f(x)}{4!} t(t-1)(t-2)(t-3) = \frac{0'0002 \cdot 2'288 (1'288)(0'288)(-0'712)}{24} \simeq 0 \right]$$

3. Y = 7'7288

↳ A) find x ; B) ERROR

* 5 sig. digits

X	3'5	3'6	3'7	3'8	3'9
Y	6'9662	7'4562	7'9830	8'5498	9'16

$\underbrace{6'9662 \quad 7'4562 \quad 7'9830}$ CLOSEST TO 7'7288

↳ for the error

A)

Y_{i1}	X_{i10}	X_{i11}	X_{i12}	X_{i3}	X_{i4}
6'9662	3'5	—	—	—	—
7'4562	3'6	0'20408	—	—	—
7'983	3'7	0'18983	-0'014015	—	—
8'5498	3'8	0'17643	-0'012253	0'0011127	—
9'16	3'9	0'16388	-0'010663	0'000933208	-0'0046466

$$X = 3'5 + (Y - 6'9662)\left[0'20408 + (Y - 7'4562)\left(-0'014015 + 0'0011127\,(\underbrace{Y - 7'983}_{-0'2542})\right)\right]$$

y = 7'7288

-0'00028285

-0'014298

-0'0038976

0'20018

0'15146

3'65115

$$\boxed{Y^{-1}(7'7288) = X = 3'65115}$$

B) $e(7'7288) = -0'0046466 \cdot \underset{(-0'821)}{(Y - 8'5498)}\,\underset{(-0'2542)}{(Y - 7'983)}\,\underset{(0'2726)}{(Y - 7'4562)}\,\underset{(0'7626)}{(Y - 6'9662)} = -0'000201593$

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1.

A) Starting fom the definition of the Chebyshev pol. of order n, and using the method of indet. coff. (use the standard base), Obtain the $\underbrace{\text{two-node}}_{n=2}$ Gauss-Chebyshev quadrature rule

* Apply $t = \cos(\theta)$

$$T_n(t) = \cos(n\theta) = \cos(n\,\mathrm{acos}(t))$$

$$T_2(t) = \cos(2\theta) = 2\cos^2\theta - 1 = 2\underbrace{\cos^2(\mathrm{acos}(t))}_{t^2} - 1 = 2t^2 - 1 = 0 \rightarrow \boxed{t_0 = \sqrt{2}/2} \quad \boxed{t_1 = -\sqrt{2}/2}$$

$$W_0 \cdot f(\sqrt{2}/2) + W_1(-\sqrt{2}/2) = 0 \longrightarrow W_1 = W_0 = \frac{\pi}{2}$$

$$\hookrightarrow \boxed{Q = \frac{\pi}{2} f\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi}{2} f\left(-\frac{\sqrt{2}}{2}\right)}$$

B) N? (pol. degree), what it means and using : $E = K f^{N+1)}(\xi)$ obtain it.

$$\int_0^{\pi} \cos^4\theta \, d\theta = 3\pi/8$$

$$nn = \text{EVEN} \longrightarrow \boxed{N = 2n+1 = 3}$$

$$n = nn - 1 = 1$$

$$\cos^4\theta = \cos^4(\mathrm{acos}(t)) = t^4$$

$$\int_0^{\pi} \cos^4\theta \, d\theta = \int_{-1}^{1} t^4 dt = \frac{3\pi}{8} = \frac{\pi}{\cancel{2}}\left(\left(\frac{\sqrt{2}}{2}\right)^4 \cdot \cancel{2}\right) + E = \frac{\pi}{4} + E = \frac{\pi}{4} + K f^{4)}(\xi)$$

$$f^{4)} = 4! = 24 \quad \frac{3\pi}{8} - \frac{\pi}{4} = 24K \rightarrow K = \frac{\pi}{8 \cdot 24} = \frac{\pi}{192} \longrightarrow \boxed{E = \frac{f^{4)}(\xi)\,\pi}{192}}$$

c) $f'(z)$ using: $x_0 = z - 3h$, $x_1 = z$; E?

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

Salou
Tarr
Cambrils

$$P_1(x) = \underbrace{\frac{(x - z)}{-3h}}_{L_0(x)} f(z - 3h) + \underbrace{\frac{(x - z + 3h)}{3h}}_{L_1(x)} f(z)$$

- $L_0'(x) = \frac{1}{-3h} \longrightarrow L_0'(z) = -1/3h$
- $L_1'(x) = \frac{1}{3h} \longrightarrow L_1'(z) = 1/3h$

$$\boxed{D = -\frac{1}{3h} f(z - 3h) + \frac{1}{3h} f(z) = \frac{f(t) - f(z - 3h)}{3h}}$$

$$\overset{n=1 \quad k=1}{} \qquad \overset{n=1 \quad k=0}{}$$

$$E = \left(f[x_0, x_1, z]\, \Pi(z)\right)' = f'[x_0, x_1, z]\, \Pi(z) + f[x_0, x_1, z]\, \Pi'(z) = f[x_0, x_1, z, z]\, \Pi(z) + f[x_0, x_1, z]\, \Pi'(z) =$$

$$= \left\{ \frac{f^{n+1+k_1)}(\xi_1)}{(n+1+k_1)!} \Pi(z) + \frac{f^{n+1+k_2)}(\xi_2)}{(n+1+k_2)!} \Pi'(z) \right\} = \frac{f^{3)}(\xi_1)}{3!} \Pi(z) + \frac{f^{2)}(\xi_2)}{2!} \Pi'(z)$$

$$\begin{cases} \Pi(z) = (x - x_0)(x - x_1) = (z - z + 3h)(z - z) = 0 \\ \Pi'(z) = (x - x_0) + (x - x_1) = (z - z + 3h) + (z - z) = 3h \end{cases}$$

$$\boxed{E = \frac{f^{2)}(\xi) \cdot 3h}{2} = \frac{3 f^{2)}(\xi) h}{2}}$$

$$\hookrightarrow \boxed{f'(z) = \frac{f(z) - f(z - 3h)}{3h} + \frac{3 f^{2)}(\xi) h}{2}}$$

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2020. ORD. P2.

2. A) $a = 1'1288$

number of subintervals (M) guaranteeing that the enor mad whe estimating $I = \int_a^2 x^2 \ln(x)\,dx$ with the compaund simpson does not exced 10^{-4} abs value

$$I = \int_{1'1288}^{2} x^2 \ln(x)\,dx$$

compaund simpson: $Q = \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n\right]$

$$E_s = -\frac{f^{4}(\xi)\cdot h^5}{90} \quad \left\{ M = \frac{(b-a)}{2h} \right\} \rightarrow E_c = \Sigma -\frac{f^{4}(\xi)\,h^5}{90} = \frac{-h^5}{90}\cdot M\,\frac{\Sigma f^{4)}(\xi)}{M}$$

$$\hookrightarrow E_c = -\frac{h^5\cdot(b-a)/2h}{90}\cdot\overline{f^{4)}}(\xi) = -\frac{h^4(b-a)}{180}\cdot\overline{f^{4)}}(\xi)$$

$$f(x) = x^2\ln(x) \longrightarrow f'(x) = 2x\ln(x) + x \rightarrow f''(x) = 2\ln(x) + 2 + 1 \rightarrow f'''(x) = 2x^{-1} \rightarrow f^{4)}(x) = -2/x^2$$

$$\hookrightarrow \begin{cases} f(1'1288) = -1'569625314 \\ f(2) = -0'5 \end{cases} \rightarrow \xi \text{ is neg} \rightarrow \text{UPPER BOUND}: 1'569625314$$

$$|E_c| = \left| -\frac{h^4(b-a)}{180}\,\overline{f^{4)}}(\xi) \right| \leq 10^{-4} \rightarrow \frac{h^4(2-1'1288)}{180}\cdot 1'569625314 \leq 10^{-4}$$

$$h^4 \leq \frac{10^{-4}\cdot 180}{0'8712\cdot 1'569625314} = 0'013163114 \longrightarrow h \leq 0'33871908 \longrightarrow M = \frac{0'8712}{2\cdot 0'33871908} = 1'28602$$

$$\hookrightarrow \boxed{M = 2}$$

B) Estimate I with 5 dec.

$$M = 2 \longrightarrow h = \frac{0'8712}{2 \cdot 2} = 0'2178$$

$$\begin{cases} x_0 = 1'1288 \\ x_1 = 1'3466 \\ x_2 = 1'5644 \\ x_3 = 1'7822 \\ x_4 = 2 \end{cases}$$

$$Q = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right] = \frac{0'2178}{3}\left[0'1543745976 + 2'772588722 + 9'5 \quad 1 + 2'1903874458 \right] =$$

$$= \boxed{1'06122}$$

C) $\int x^2 \ln(x)\, dx = x^3(-1+3\ln x)/9 \quad + C$ error before?

$$\text{EXACT}: \; I = \left. \frac{x^3(-1+3\ln x)}{9} \right]_{1'1288}^{2} = 0'959503592 + 0'101725663 = 1'06123$$

$$E = \text{EXACT} - \text{APROX} = \boxed{0'00001}$$

3. $\theta_0 = 0'52288$

$\theta'' + 2\theta' + 2\sin(\theta) = 0$ $\theta(0) = \theta_0$, $\theta'(0) = 0$

A) England's Method:

$$y_{k+1} = y_k + \frac{h}{6}(k_1 + 4k_3 + k_4) \qquad \begin{cases} k_1 = f(x_k, y_k) \\ k_2 = f\left(x_k + \frac{h}{2}, \; y_k + \frac{h}{2}k_1\right) \\ k_3 = f\left(x_k + \frac{h}{2}, \; y_k + \frac{h}{4}k_2\right) \\ k_4 = f\left(x_k + h, \; y_k - h k_2 + 2hk_3\right) \end{cases}$$

Take 1 step $\rightarrow t = 0'1 \longrightarrow h = 0'1$

*7 sign. digits

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3. A) $\Theta_{k+i} = \Theta_k + \dfrac{(k_1 + 4k_3 + k_4)}{6}$

$$\begin{cases} k_1 = h_k \cdot f(t_k, \Theta_k) \\ k_2 = h_k \cdot f(t_k + h_k/2, \Theta_k + k_1/2) \\ k_3 = h_k \cdot f(t_k + h_k/2, \Theta_k + k_2/4) \\ k_4 = h_k \, f(t_k + h_k, \Theta_k - k_2 - 2k_3) \end{cases}$$

$$\underset{\sim}{\Theta} = \begin{pmatrix} \Theta \\ \Theta' \end{pmatrix} \longrightarrow f(t,\Theta) = \begin{pmatrix} \Theta' \\ \Theta'' \end{pmatrix} = \begin{pmatrix} \Theta' \\ -2(\Theta' + \sin(\Theta)) \end{pmatrix}$$

$\boxed{k=0}$ $\quad \Theta = 0, \quad \underset{\sim}{\Theta_0} = \begin{pmatrix} 0{'}52288 \\ 0 \end{pmatrix}, \quad h = 0{'}1$

$$k_1 = 0{'}1 \cdot \begin{pmatrix} 0 \\ -0{'}9987548 \end{pmatrix} = \begin{pmatrix} 0 \\ -0{'}09987548 \end{pmatrix}$$

$$k_2 = 0{'}1 \cdot f\left(0{'}05, \begin{pmatrix} 0{'}52288 \\ -0{'}04993774 \end{pmatrix}\right) = 0{'}1 \begin{pmatrix} -0{'}04993774 \\ -0{'}898879305 \end{pmatrix} = \begin{pmatrix} -0{'}004993774 \\ -0{'}08988793 \end{pmatrix}$$

$$k_3 = 0{'}1 \, f\left(0{'}05, \begin{pmatrix} 0{'}521631556 \\ -0{'}022471982 \end{pmatrix}\right) = 0{'}1 \begin{pmatrix} -0{'}022471982 \\ -0{'}95164678 \end{pmatrix} = \begin{pmatrix} -0{'}0022471982 \\ -0{'}095164678 \end{pmatrix}$$

$$k_4 = 0{'}1 \, f\left(0{'}1, \begin{pmatrix} 0{'}532236817 \\ 0{'}280217286 \end{pmatrix}\right) = \begin{pmatrix} 0{'}0280217286 \\ -0{'}1575558496 \end{pmatrix}$$

$$\boxed{\Theta(0{'}1) = \begin{pmatrix} 0{'}52288 \\ 0 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} -0{'}019032935 \\ -0{'}63638378 \end{pmatrix} = \begin{pmatrix} 0{'}519707884 \\ -0{'}106063965 \end{pmatrix}}$$

B) using : $\Theta_0 = \begin{pmatrix} 0'52\overset{678}{288} \\ 0 \end{pmatrix}$, $\Theta(0'1) = \begin{pmatrix} 0'52208 \\ -0'090871 \end{pmatrix}$, $\Theta(0'2) = \begin{pmatrix} 0'50612 \\ -0'16294 \end{pmatrix}$, $\Theta(0'3) = \begin{pmatrix} 0'48686 \\ -0'2197 \end{pmatrix}$

$$Y_{k+1} = Y_{k-3} + \frac{h}{3}\left(f_{k+1} + 4f_k + f_{k-1}\right) \quad (C)$$

$t=0'4$; 1 step

$$f = \begin{pmatrix} \theta' \\ -2(\theta' + \sin\theta) \end{pmatrix}$$

$$Y_{k+1} = Y_{k-3} + \frac{4h}{3}\left(2f_k - f_{k-1} + 2f_{k-2}\right) \quad (P)$$

$t_0 = 0$, $t_1 = 0'1$, $t_2 = 0'2$, $t_3 = 0'3$, $t_4 = 0'4$ $\quad (k=3)$

$$Y_{4p} = Y_0 + \frac{4h}{3}\left(2f_3 - f_2 + 2f_1\right) = \begin{pmatrix} 0'52678 \\ 0 \end{pmatrix} + \frac{0'4}{3}\begin{pmatrix} -0'4394 + 0'16294 - 0'181742 \\ -0'99261 + 0'64369 - 1'63126 \end{pmatrix} = \boxed{\begin{pmatrix} 0'465864 \\ -0'264024 \end{pmatrix}}$$

$$Y_{4C1} = Y_2 + \frac{h}{3}\left(f_p + 4f_3 + f_2\right) = \begin{pmatrix} 0'50612 \\ -0'16294 \end{pmatrix} + \frac{0'1}{3}\begin{pmatrix} -0'264024 - 0'8788 - 0'16294 \\ -0'37034 - 1'98522 - 0'64369 \end{pmatrix} = \boxed{\begin{pmatrix} 0'46259 \\ -0'26291 \end{pmatrix}}$$

4. $$f''(z) = \frac{-f(z-2h) + 16f(z-h) - 30f(z) + 16f(z+h) - f(z+2h)}{12h^2} + \frac{f^{6}(\xi)h^4}{90}$$

A)

$\begin{cases} z = 1'5678 \\ \varepsilon = 1'678 \cdot 10^{-6} \end{cases}$ Estimate h_{opt} ; $\begin{cases} f(x) = \ln(x) + \cos(x) \\ M = |f^{6)}(z)| \text{ upperbound} \end{cases}$

$f(x) = \ln(x) + \cos(x) \rightarrow f' = 1/x - \sin(x) \rightarrow f'' = -1/x^2 - \cos(x) \rightarrow f''' = 2/x^3 + \sin(x)$

$f^{IV)} = -6/x^4 + \cos(x) \rightarrow f^{V)} = 24/x^5 - \sin(x) \rightarrow f^{VI)} = -120/x^6 - \cos(x)$

$f^{6)}(1'5678) = -8'083478661$

1). $|E| = \left|\frac{f^{6)}(z)\,h^4}{90}\right| \leq \frac{Mh^4}{90} = 0'089816429h^4 = g_1(h)$

2). $|AF| = \sum_{i=0} |A_i| = \frac{1+16+30+16+1}{12h^2} = \frac{1}{0'1875h^2}$

3.) $|E_r| = \varepsilon|AF| = \frac{1'678\cdot 10^{-6}}{0'1875h^2} = g_2(h)$

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4. A)

$$4)\ g(h) = g_1(h) + g_2(h) = 0'089816429\, h^4 + \frac{1'678\cdot 10^{-6}}{0'1875 h^2}$$

$$5.)\ g'(h) = 0'359265718\, h^3 - \frac{3'356\cdot 10^{-6}}{0'1875\, h^3} = 0$$

$$\hookrightarrow 0'35899376 h^6 = \frac{3'356\cdot 10^{-6}}{0'1875} \longrightarrow \boxed{h_{opt} = 0'191849272}$$

B) Estimate $f''(z)$ using $h_{opt} = 0'19185$

$f(z-2h) = f(1'1841) = 0'54611$

$f(z-h) = f(1'37595) = 0'51276$

$f(z) = f(1'5678) = 0'45267$

$f(z+h) = f(1'75965) = 0'37738$

$f(z+2h) = f(1'9515) = 0'29702$

$$\boxed{D = \frac{-0'54611 + 16\cdot 0'51276 - 30\cdot 0'45267 + 16\cdot 0'37738 - 0'29702}{12\cdot 0'19185^2} = -0'40978}$$

$$\boxed{E = \frac{f^{(6)}(\xi)\cdot h^4}{90} = -0'089816 \cdot 0'19185^4 = -0'00012167}$$

$$\%\left|\frac{E}{g_{min}}\right| = 100 \cdot \frac{0'00012167}{}$$

2020. EXTRA

30/05/21

1. $x_1 = 2'678$; 6 sign. digits

A) Calculate the osculating. pol. of $f(x) = \sin(x)$ by the nodes $x_i = \{0, x_1, 2\pi\}$

Coincides with 1st der. at $x_0 = 0$ and at $x_2 = 2\pi$

at $x_1 = 2'678$ only $f(x)$

$$f(x) = \sin(x) \longrightarrow f'(x) = \cos(x) \begin{cases} \nearrow f'(0) = 1 \\ \searrow f'(2\pi) = 1 \end{cases}$$

x_i	h_{i0}	h_{i1}	h_{i2}	h_{i3}	f_{i4}
0	0	—	—	—	—
0	0	1	—	—	—
2'678	0'447164	0'166977	-0'311062	—	—
2π	0	-0'124034	-0'0463158	0'0421357	—
2π	0	1	0'311783	0'0569932	0'00236464

$$\boxed{P_4(x) = (x) - 0'311062\,x^2 + 0'0421357\,x^2(x-2'678) + 0'00236464\,x^2(x-2'678)(x-2\pi)}$$

B) Evaluate it optimally at $x = \pi/2$ and calculate E_r

$$P_4 = x\left[1 + x\left(-0'311062 + (x-2'678)\left(0'041357 + 0'00236464\,\underbrace{(x-2\pi)}\right)\right)\right]$$

$x = \pi/2$

-4'71239

-0'0111431

0'0302139

-0'0334529

-0'3445149

-0'54163

0'458837

0'720739

$\rightarrow \boxed{P_3(\pi/2) = 0'72039}$

$$\boxed{E = \text{EXACT} - \text{APPROX} = f(\pi/2) - 0'72039 = 0'27961}$$

c). Evaluate the nat. cubic spline $s(x)$ by $\{0, 2'678, 2\pi\}$ at $x = \pi/2$

NAT. CUBIC SPLINE $\begin{cases} P_0''(x) = 0 = M_0 \\ P_{n-1}''(x) = 0 = M_2 \end{cases}$ $\longrightarrow$ 3 nodes, 2 cond $\longrightarrow$ MATRIX 1×1

$$T_{11} \cdot M_1 = d_{11} \; ; \quad T_{11} = 2(\overset{\overset{2'678}{\parallel}}{h_0} + h_1) = 2(0 + 2\pi) = 4\pi \longrightarrow h_0 = 2'678 \longrightarrow h_1 = 3'60519$$

(above: $x_0 + x_2$)

$$d_{11} = 6\left(\frac{y_2 - y_1}{h_1} - \frac{y_1 - y_0}{h_0}\right) = 6\left(\frac{\overset{0}{\cancel{\sin(2\pi)}} - \sin(2'678)}{3'60519} - \frac{\sin(2'678) - \overset{0}{\cancel{\sin(0)}}}{2'678}\right) = -6 \cdot 0'447164 \cdot 0'650791 =$$

$$= 1'74606$$

$$M_1 = \frac{-1'74607}{4\pi} = -0'138947$$

? $$P_0'(x_0) = \frac{y_1 - y_0}{h_0} - 0 - (M_1 - 0)\frac{h_0}{6} = +0'166977 + \frac{2'67 \cdot 0'138947}{6} = 0'228994$$

$$s(\pi/2) = P_0(x = \pi/2) = P_0(x_0) + P_0'(x_0)(x - x_0) + M_0\frac{(x - x_0)^2}{2} + \frac{M_1 - M_0}{h_0}\frac{(x - x_0)^3}{6}$$

$$\boxed{s(\pi/2) = 0 + (\pi/2 - 0) \cdot 0'228994 + 0 + \frac{(-0'138947)}{2'678}\frac{(\pi/2)^3}{6} = 0'326187}$$

D) E

$$\boxed{E = f(\pi/2) - s(\pi/2) = 0'673813}$$

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2020 EXTRA.

2.

A) Obtain an interp. quadr. of the form: $\int_0^{3h} f(x)dx \simeq Q(f) = A_0 f(0) + A_1 f(h) + A_2 f(3h)$

$R(f) = K f^{(3+1)}(\xi)$

$\cdot f(x) = 1 \longrightarrow \int_0^{3h} 1 dx = 3h = A_0 \cdot 1 + A_1 \cdot 1 + A_2 \cdot 1 \rightarrow \boxed{A_0 + A_1 + A_2 = 3h}$ (1)

$\cdot f(x) = x \longrightarrow \int_0^{3h} x dx = \frac{9h^2}{2} = 0 + A_1 h + A_2 3h \rightarrow \boxed{A_1 + 3A_2 = 9h/2}$ (2)

$\cdot f(x) = x^2 \longrightarrow \int_0^{3h} x^2 dx = \frac{3^3 h^3}{3} = 9h^3 = 0 + A_1 h^2 + 9A_2 h^2 \rightarrow \boxed{A_1 + 9A_2 = 9h}$ (3)

(2) - (3)

$$\begin{cases} A_1 = 9h/2 - 3A_2 \\ \hookrightarrow \frac{9h}{2} - 3A_2 + 9A_2 = 9h \rightarrow 6A_2 = 9h - \frac{9h}{2} = \frac{9h}{2} \rightarrow \boxed{A_2 = \frac{3h}{4}} \rightarrow \boxed{A_1 = \frac{9h}{2} - \frac{9h}{4} = \frac{9h}{4}} \end{cases}$$

$$\boxed{A_0 = 3h - \frac{3h}{4} - \frac{9h}{4} = \frac{12h - 3h - 9h}{4} = 0}$$

$$\boxed{Q = \frac{3h}{4}\left[3f(h) + f(3h)\right]}$$

$\cdot f(x) = x^3 \longrightarrow \int_0^{3h} x^3 dx = \frac{81h^4}{4} = \frac{3h}{4}\left[3h^3 + 27h^3\right] + E \rightarrow E \neq 0$

$f^{3)}(x) = 3! = 6 \longrightarrow \frac{81h^4}{4} = \frac{90h^4}{4} + 6K \longrightarrow 6K = \frac{-9h^4}{4} \rightarrow \boxed{K = -\frac{3h^4}{8}} \rightarrow \boxed{E = -\frac{f^{4)}(\xi)\, 3h^4}{8}}$

B) For mulas for any $[a,b]$ interval ?

* change of variable

$0 \longrightarrow a$

$3h \longrightarrow b$

$$\longrightarrow \boxed{I = \int_a^b f(x)\,dx = \int_0^{(b-a)=3h} f(x+a)\,dx =}$$

$$\boxed{= \frac{3h}{4}\left[3\cdot f(a+h) + f(a+3h)\right] - \frac{3}{8}h^4 f^{3)}(\xi+a) \quad fs \quad \xi \in (0,3h)}$$

C) Compound formula M times in the interval

$$\begin{cases} h=(b-a)/3M \\ a_i = a+(i-1)3h \\ b_i = a+i3h \end{cases}$$

$$h = \frac{(b-a)}{3M}$$

$$b_i = a_i+3h$$

$$\boxed{Q = \frac{3h}{4}} \Sigma\left(3f(a_i+h) + f(a_i+3h)\right) = \frac{3h}{4} \cdot \Sigma\left(3f(a_i+h)+f(b_i)\right) =$$

$$= \frac{3(b-a)}{3M\cdot 4} \Sigma\left(3f(a_i+h)+f(b_i)\right) = \boxed{\frac{(b-a)}{4M} \Sigma\left(3f(a_i+h)+f(b_i)\right)}$$

$$\boxed{E = \Sigma\left(-\frac{3}{8}h^4 f^{3)}(\xi+a_i)\right) = -\frac{3h^4}{8}\cdot M \cdot \frac{\Sigma f^{3)}(\xi+a_i)}{M} = -\frac{\cancel{3}\cdot h^{\cancel{4}\,3}}{8}\frac{(b-a)}{\cancel{3M}}\overline{f^{3}}(\xi+a_i) = \frac{h^3(b-a)}{8} + \overline{f^{3)}}(\xi+a_i)}$$

D) $r = 1'678$; $\int_r^5 x^2 \ln(x)\,dx$ $M=2$; find upperbound error

31/05/21

2020. EXTRA

2 c)

$$h = \frac{(5-1'678)}{3\cdot 2} = 0'55367$$

$$x_i = \{1'678+h,\ 1'678+3h,\ 5-2h,\ 5\} = \{2'2317,\ 3'339,\ 3'8927,\ 5\}$$

$$W_i = \frac{3h}{4}\cdot\{3,\ 1,\ 3,\ 1\} = \{1'245..$$

3.

$$\begin{cases} y''(x) = 3+5x^2-y(x)-xy'(x) \\ y(0)=1\ ,\ y'(0)=0 \end{cases}$$

use to approx : $y(0'25)$, $y'(0'25)$

$$y_{k+1} = y_k + \frac{(k_1+3k_3)}{4} \qquad \begin{cases} k_1 = h_k\cdot f(x_k, y_k) \\ k_2 = h_k\, f(x_k + h_k/3,\ y_k + k_1/3) \\ k_3 = h_k\cdot f(x_k + 2h_k/3,\ y_k + 2k_2/3) \end{cases}$$

1 step, 6 significant digits ; with $h=0'125 \longrightarrow$ how would the error decrease?

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} \longrightarrow \underset{\sim}{y}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ 3+5x^2-y-xy' \end{pmatrix} = f(x,y)$$

. 1 STEP to reach $\underset{\sim}{y}(0'25) \longrightarrow h=0'25$

$\boxed{k=0}$ $\quad y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $x=0$

$$k_1 = 0'25\cdot f\left(0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 0'25\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 0'5 \end{pmatrix}}$$

$$k_2 = 0'25\, f\left(0'0833333, \begin{pmatrix} 1 \\ 0'166667 \end{pmatrix}\right) = 0'25\begin{pmatrix} 0'166667 \\ 2'02083 \end{pmatrix} = \boxed{\begin{pmatrix} 0'0416668 \\ 0'502508 \end{pmatrix}}$$

$$k_3 = 0'25\cdot f\left(0'166667, \begin{pmatrix} 1'02778 \\ 0'335005 \end{pmatrix}\right) = 0'25\begin{pmatrix} 0'335005 \\ 2'05528 \end{pmatrix} = \boxed{\begin{pmatrix} 0'0837513 \\ 0'51382 \end{pmatrix}}$$

$$\underset{\sim}{Y}(0'25) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0'25 \begin{pmatrix} 0 + 3 \cdot 0'0837513 \\ 0'5 + 3 \cdot 0'51382 \end{pmatrix} = \begin{pmatrix} 1'06281 \\ 0'510365 \end{pmatrix} = \begin{pmatrix} y(0'25) \\ y'(0'25) \end{pmatrix}$$

* 3 EVALUATIONS → OC3 → if $h = 0'25$ is considered small enough: $e_{h=0'125} = \left(\frac{0'125}{0'25}\right)^3 = 0'125$ compare

4

Adams - Moulton (4) abs. stability Region: $(-3,0)$

would it be adequate to solve : $\begin{cases} u_1' = 998 u_1 + 1998 u_2 \\ u_2' = -999 u_1 - 1999 u_2 \end{cases}$ with $h = 0'1$

$$J = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix}$$

$$|J - \lambda I| = 0 \rightarrow \begin{vmatrix} 998-\lambda & 1998 \\ -999 & -1999-\lambda \end{vmatrix} = -(1999+\lambda)(998-\lambda) + 1998 \cdot 999 = 0$$

$$\hookrightarrow -1996002 + 1995002 - 1001\lambda - \lambda^2 = 0 = \lambda^2 + 1001\lambda + 1000 = 0$$

$$\lambda = \frac{-1001 \pm \sqrt{998001}}{2} = \frac{-1001 \pm 999}{2} \begin{matrix} \nearrow \lambda_1 = -1000 \\ \searrow \lambda_2 = -1 \end{matrix} \Big\} < 0 \rightarrow$$

STABLE SYSTEM BUT VERY DISPROPORTIONATE

↳ STIFF

- $\text{Eigs}(hJ) = h \cdot \text{Eigs}(J) = 0'1 \cdot \{-1000, -1\} = \{-100, -0'1\} \neq \{-3, 0\}$

31/05/21

2020. EXTRA.

5 $f''(z)$; z, $z+h$, $z+2h$ and the error.

TAYLOR ("AD-HOC")

$$2\left(\cdot\ f(z+h) = \frac{f(z)}{0!}h^0 + \frac{f'(z)h}{1!} + \frac{f''(z)h^2}{2!} + \frac{f^{3)}(\xi_1)h^3}{3!} \right)$$

$$- \quad \cdot\ f(z+2h) = \frac{f(z)}{0!}(2h)^0 + \frac{f'(z)2h}{1!} + \frac{f''(z)\cdot 4h^2}{2!} + \frac{f^{3)}(\xi_2)8h^3}{3!}$$

$$2f(z+h) - f(z+2h) = f(z) - f''(z)h^2 + \underbrace{\frac{f^{3)}(\xi_1)h^3}{3} - \frac{4f^{3}(\xi_2)}{3}h^3}_{"E"}$$

$$\hookrightarrow 2f(z+h) - f(z+2h) - f(z) + E = -f''(z)h^2$$

$$\hookrightarrow \boxed{f''(z) = \underbrace{\frac{f(z+2h)+f(z)-2f(z+h)}{h^2}}_{D} + \underbrace{\frac{f^{3)}(\xi_1) - 4f^{3)}(\xi_2)}{3}h}_{E}}$$

04/05/21

2019. ORD.

1 Prove that $p(x) \in \mathbb{P}_n$ interpolating a function $f(x)$ in the simple nodes $x_0, x_1, \dots, x_n$ is unique

UNIQUENESS : Reduction to absurdity

we assume $p_n(x)$, $q_n(x)$ two different polynomials of DEGREE $\leq n$ passing by the interpolation points : $x_0, x_1, \dots, x_n$

$$r_n(x) = p_n(x) - q_n(x) \in \mathbb{P}_n$$

With $n+1$ roots (the nodes x_i) we will contradict the Fundamental Theorem of Algebra unless $r_n(x) = 0 \longrightarrow$ so : $p_n(x) = q_n(x) \in \mathbb{P}_n$

which means that it is UNIQUE

2. Let $f(x)$ be a pol. of degree 5 with $f^{5)} = 144$. What is the value of its finite diff. of order 5 when the distance between nodes is 0'5? And the divided diff?

- $h = 0'5$

* DIVIDED DIFF : $$\boxed{f[x_0, x_1, \dots, x_5] = \frac{\overbrace{f^{5)}(\xi)}^{f^{n)}(\xi)}}{\underbrace{5!}_{n!}}}$$ RELATION : (DIV DIFF ⟷ DERIVATIVES)

$$\boxed{f[x_0, x_1, \dots, x_5] = \frac{144}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1'2}$$

* FINITE DIFF : $$\boxed{f[x_0, x_1, \dots, x_5] = \frac{\overbrace{\Delta^5 f(x_0)}^{\Delta^n f(x_0)}}{\underbrace{5!\, h^5}_{n!\, h^n}} = \frac{\overbrace{f^{5)}(\xi)}^{f^{n)}(\xi)}}{\underbrace{5!}_{n!}}}$$

$$\frac{\Delta^5 f(x_0)}{0'5^5} = 144 \longrightarrow \boxed{\Delta^5 f(x_0) = 4'5}$$

3.

$$p(x) = \boxed{3} + \boxed{\frac{1}{2}}\overset{\overset{1}{\uparrow}}{(x-1)} + \boxed{\frac{1}{3}}(x-1)\overset{\overset{1'5}{\uparrow}}{(x-1'5)} \boxed{-2}(x-1)(x-1'5)\overset{\overset{0}{\uparrow}}{x}$$

$$q(x) = \frac{5}{3} - \frac{2}{3}(x-2) - \frac{5}{3}(x-2)x - 2\underset{\underset{2}{\downarrow}}{(x-2)}x(x-1'5)$$

$\Big\} \; ⊜$

Two representation of the interp. pol of f(x) on the same four nodes: x_0, x_1, x_2, x_3

Obtain the table of divided diff of p(x) without evaluating the pol.

x	$f_{i,0}$	$f_{i,1}$	$f_{i,2}$	$f_{i,3}$
1	$\boxed{3}$	—	—	—
1'5	13/4	$\boxed{1/2}$	—	—
0	3	1/6	$\boxed{1/3}$	—
2	5/3	-2/3	-5/3	$\boxed{-2}$

$$f_{1,1} = \frac{f_{1,0} - 3}{1'5 - 1} = 1/2 \longrightarrow f_{1,0} = \frac{1}{4} + 3 = 13/4$$

$$f_{2,2} = \frac{f_{2,1} - 0'5}{0 - 1} = 1/3 \longrightarrow f_{2,1} = -\frac{1}{3} + \frac{1}{2} = 1/6$$

$$f_{3,3} = \frac{f_{3,2} - 1/3}{2 - 1} = -2 \longrightarrow f_{3,2} = -2 + \frac{1}{3} = -5/3$$

$$f_{2,1} = \frac{f_{2,0} - 13/4}{-1'5} = \frac{1}{6} \longrightarrow f_{2,0} = \frac{13}{4} - \frac{1}{4} = \frac{12}{4} = 3$$

$$f_{3,2} = \frac{f_{3,1} - 1/6}{2 - 1'5} = \frac{-5}{3} \longrightarrow f_{3,1} = -\frac{5}{6} + \frac{1}{6} = -\frac{4}{6} = -2/3$$

$$f_{3,1} = \frac{f_{3,0} - 3}{2} = -2/3 \longrightarrow f_{3,0} = -\frac{4}{3} + 3 = 5/3$$

04/05/21

2019. ORD.

4 max length = 3m when $t = 5$

min length = 1'5m when $t = 1$

a) Operating optimally, use a polynomial of the lowest possible degree agreeing to those data. → Estimate the initial value.

$$\begin{cases} L(5) = 3m \rightarrow L'(5) = 0 \rightarrow 0/1! = 0 \\ L(1) = 1'5m \rightarrow L'(1) = 0 \rightarrow 0/1! = 0 \end{cases}$$

z_i	$f_{i,0}$	$f_{i,1}$	$f_{i,2}$	$f_{i,3}$
5	$\boxed{3}$	—	—	—
5	3	$\boxed{0}$	—	—
1	1'5	0'375	$\boxed{-0'09375}$	—
1	1'5	0	0'09375	$\boxed{-0'046875}$

$$P(x) = 3 + 0(t-5) - 0'09375 \cdot (t-5)^2 - 0'046875(t-5)^2(t-1)$$

$$\boxed{P_3(t) = 3 - 0'09375(t-5)^2 - 0'046875(t-5)^2(t-1)}$$

$t=0$

$$P(t) = 3 + (t-5)^2 \left[-0'09375 - \underbrace{0'046875(t-1)}_{-0'046875}\right]$$

$-0'046875$

$-1'171875$

$1'828125$

$$\boxed{P(0) \simeq L(0) = 1'828125}$$

b) $t=4$; $L = 2'7m$ → error?

$e(x) = P_4(x) - P_3(x)$

z_i	$f_{i,0}$	$f_{i,1}$	$f_{i,2}$	$f_{i,3}$	f_{i4}
...	...				—
4	2'7	0'4	0'1333	-0'039583	-0'007292

$$P_4(t) = 3 - 0'09375(t-5)^2 - 0'046875(t-5)^2(t-1) - 0'007292(t-5)^2(t-1)^2$$

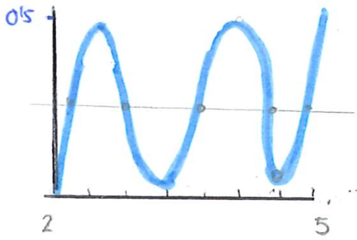
$$P_4(t) = \underbrace{3 + \underbrace{(t-5)^2 \left[\underbrace{-0'09375 + \underbrace{(t-1)\left(\underbrace{-0'046875 - \underbrace{0'007292(t-1)}_{-0'007292}}_{-0'039583}\right)}_{0'039583}}_{-0'054147}\right.}_{-1'353675}}_{1'646325} \qquad \boxed{t=0}$$

$$\boxed{e(0)} = 1'646325 - 1'828125 = \boxed{-0'1818}$$

5 What nodes x_i seem to have been used in the graph of the polynomial

$\Pi(x) = (x-x_0)(x-x_1)\cdot\ldots\cdot(x-x_n)$ over the interval $[2,5]$

Justify and use 4 or more decimals.



$\longrightarrow \underbrace{\text{CHEBYSHEV with 5 NODES}}_{\in(-1,1)}$

$$\boxed{t_i = \cos\left(\frac{\pi/2 + k\pi}{n}\right)}$$

$$\overset{(t)}{[-1,1]} \longrightarrow \overset{(x)}{[2,5]} \; ; \; x = \frac{a+b}{2} + \frac{b-a}{2}t \rightarrow \boxed{x = 3'5 - 1'5t}$$

$t_0 = \cos(0'31416) = 0'9511$

$t_1 = \cos(0'9424778) = 0'5878$

$t_2 = \cos(1'5708) = 0$

$t_3 = \cos(2'1991) = -0'5878$

$t_3 = \cos(2'82743) = -0'9511$

$x_0 = 2'0734$

$x_1 = 2'6183$

$x_2 = 3'5$

$x_4 = 4'3817$

$x_5 = 4'9266$

$$\boxed{x_i = \{2'0734, 2'6183, 3'5, 4'3817, 4'9266\}}$$

04/05/21

2019. ORD.

6. a) Obtain the interpolatory quadrature rule of the highest degree as possible with the form: $\int_0^h f(x)dx \simeq A_0 f(0) + A_1 f'(0) + A_2 f''(0) + A_3 f''(0)$

- $f(x) = 1 \longrightarrow \int_0^h 1\,dx = h = A_0 \longrightarrow \boxed{A_0 = h}$
- $f(x) = x \longrightarrow \int_0^h x\,dx = \frac{h^2}{2} = 0 \cdot h + A_1 \longrightarrow \boxed{A_1 = h^2 12}$
- $f(x) = x^2 \longrightarrow \int_0^h x^2 dx = \frac{h^3}{3} = 2A_2 \longrightarrow \boxed{A_2 = h^3/6}$
- $f(x) = x^3 \longrightarrow \int_0^h x^3 dx = \frac{h^4}{4} = 32A_3 \longrightarrow \boxed{A_3 = h^4/24}$

$$\boxed{\int_0^h f(x) \simeq h f(0) + \frac{h^2}{2} f'(0) + \frac{h^3}{6} f''(0) + \frac{h^4}{24} f'''(0) \quad = \Phi}$$

b) obtain its pol. degree and the expression of its truncation error (you can operate as if it was a $\underbrace{\text{Newton-Cotes}}_{\text{EQUALLY SPACED NODES}}$ formula)

- $f(x) = x^4 \longrightarrow \int_0^h x^4 dx = \frac{h^5}{5} \neq 0 + 0 + 0 + 0 \longrightarrow \boxed{\text{POLYNOMIAL DEGREE} = 3}$

$E = K f^{4)}(\xi)$

$f' = 4x^3 \rightarrow f'' = 12x^2 \rightarrow f''' = 24x \rightarrow f^{IV} = 24$

$\frac{h^5}{5} = 0 + K \cdot 24 \longrightarrow K = \frac{h^5}{120} \longrightarrow \boxed{E = \frac{h^5 \cdot f^4(\xi)}{120}}$

c) change of variable to allow the formula to be applied over any int. $[a,b]$

$[0,h] \rightarrow [a,b]$

$x = a + \frac{(b-a)}{h} t$

$$\int_a^b f(x)\,dx = \int_0^h f\left(a + \frac{(b-a)}{h} t\right) \cdot \frac{(b-a)}{h}\,dt = \int_0^h g(t)\,dt$$

$$g(t) = f\left(a + \frac{(b-a)}{h} t \right) \cdot \frac{(b-a)}{h}$$

- $g(0) = (b-a)/h$
- $g'(0) = \frac{(b-a)}{h} \cdot \frac{(b-a)}{h} = (b-a)^2/h^2$
- $g''(0) = \frac{(b-a)}{h} \; \frac{(b-a)^2}{h} = (b-a)^3/h^3$
- $g'''(0) = \frac{(b-a)}{h} \cdot \frac{(b-a)^3}{h} = (b-a)^4/h^4$

$$\boxed{Q = (b-a) f(a) + \frac{(b-a)^2}{2} f'(a) + \frac{(b-a)^3}{6} f''(a) + \frac{(b-a)^4}{24} f'''(a)}$$

$$\boxed{E = \frac{h^5 f^{4)}(\xi)}{120} = \frac{\cancel{h^5} \cdot (b-a)^5/\cancel{h^5} \cdot f^{4)}(\xi)}{120} = \frac{(b-a)^5 f^{4)}(\xi)}{120}}$$

$$\boxed{I = (b-a) f(a) + \frac{(b-a)^2}{2} f'(a) + \frac{(b-a)^3}{6} f''(a) + \frac{(b-a)^4}{24} f'''(a) + \frac{(b-a)^5 f^{4)}(\xi)}{120}}$$

d) Obtain the previous expression but compose it M times.

$\boxed{h = \frac{(b-a)}{M}}$ $a_i = a + (i-1)h$; $b_i = a + ih$ $i = 1, 2, \dots, M$

$$Q = \sum_{i=1}^{M} Q_i = \sum_{i=1}^{M} \left[(b_i - a_i) f(a_i) + \frac{(b_i-a_i)^2}{2} f'(a_i) + \frac{(b_i-a_i)^3}{6} f''(a_i) + \frac{(b_i-a_i)^4}{24} f'''(a_i) \right] =$$

$* \; b_i - a_i = \cancel{a} + ih - \cancel{a} - (i-1)h = h$

$$Q = \sum_{i=1}^{M} \left[h f(a+(i-1)h) + \frac{h^2}{2} f'(a+(i-1)h) + \frac{h^3}{6} f''(a+(i-1)h) + \frac{h^4}{24} f'''(a+(i-1)h) \right] =$$

$$\boxed{= \sum_{i=0}^{M-1} \left[h f(a+ih) + \frac{h^2}{2} f'(a+ih) + \frac{h^3}{6} f''(a+ih) + \frac{h^4}{24} f'''(a+ih) \right] \quad \text{WHERE } h = (b-a)/M}$$

$$E = \sum_{i=1}^{M} E_i = \sum_{i=1}^{M} \frac{(b_i-a_i)^5 f^{4)}(\xi)}{120} = \frac{h^5}{120} \sum_{i}^{M} f^{4)}(\xi) = \frac{h^5}{120} \cdot M \frac{\sum^{M} f_i^{4)}(\xi)}{M} = \frac{h^4 (b-a)}{120 M} \sum_{i=1}^{M} f^{4)}(\xi)$$

$$\boxed{E = \frac{h^4 (b-a)}{120} f^{4)}(\xi) \qquad \xi \in [a,b]}$$

05/04/21

2019. ORD.

6 e) $M=2$; $\int_2^3 \cos(2x)\,dx$

$h = \frac{(3-2)}{2} = 1/2$; $a+ih$ $\begin{cases} i=0 \rightarrow 2 \\ i=1 \rightarrow 5/2 \end{cases}$

$f(x)=\cos(2x) \rightarrow f'(x) = -2\sin(2x) \rightarrow f''(x) = -4\cos(2x) \rightarrow f'''(x) = 8\sin(2x) \rightarrow f^{(iv)}(x) = 16\cos(2x)$

$$\boxed{Q} = \frac{1}{2}\left(\cos(4)+\cos(5)\right) + \frac{1}{\cancel{8}4}(-\cancel{2})\cdot\left(\sin(4)+\sin(5)\right) + \frac{(-\cancel{4})}{\underset{12}{\cancel{48}}}\left(\cos(4)+\cos(5)\right) + \frac{\cancel{8}}{\underset{48}{3\cancel{8}4}}\left(\sin(4)+\sin(5)\right) =$$

$$= \frac{5}{12}\left(\cos(4)+\cos(5)\right) - \frac{11}{48}\left(\sin(4)+\sin(5)\right) = \boxed{0'239028453}$$

$$E = \frac{(\cancel{1/2})^{\cancel{4}}\cdot(3-2)}{120}\cdot\cancel{16}\cos(2x) = \frac{\cos(2x)}{120} \longrightarrow \boxed{x=3} \rightarrow \boxed{|E| \leq 0'008001419}$$

M10/5/21

T 7. a) using Taylor series, obtain the num. diff. formula of the highest order to estimate the 2nd der: $f''(z)$ from $f(z), f(z-h)$ and $f(z+h)$

- $f(z+h) = f(z) + \frac{f'(z)\cdot h}{1!} + \frac{f''(z)\cdot h^2}{2!} + \frac{f'''(z)h^3}{3!} + \frac{f^{(4)}(\xi_1)h^4}{4!}$ f.s $\xi_1 \in (z,z+h)$ ∧ $f \in C^3$

$\oplus$

- $f(z-h) = f(z) - \frac{f'(z)h}{1!} + \frac{f''(z)h^2}{\underset{''2}{2!}} - \frac{f'''(z)h^3}{\underset{''6}{3!}} + \frac{f^{(4)}(\xi_2)h^4}{4! = 24}$ f.s. $\xi_2 \in (z-h,z)$ ∧ $f \in C^3$

$$f(z+h)+f(z-h) = 2f(z) + 0 + f''(z)h^2 + 0 + \frac{f^{(4)}(\xi_1)+f^{(4)}(\xi_2)}{24}h^4$$

$$\hookrightarrow f''(z) = \underbrace{\frac{f(z+h)+f(z-h)-2f(z)}{h^2}}_{D} - \underbrace{\frac{f^{(4)}(\xi_1)+f^{(4)}(\xi_2)}{24}h^2}_{E}$$

$$\hookrightarrow \boxed{D = \frac{f(z+h)+f(z-h)-2f(z)}{h^2}} \qquad \boxed{E = -\frac{f^{4}(\xi)}{12}h^2 \quad \text{f.s } \xi \in (z-h,z+h)}$$

b) Derive the general expression of the trunc. error. of the estimation of $f''(z)$ by differenciating the interpolation error. (particularize it for the case above).

$$\cdot\ E = e''(z) = \left(f[x_0, x_1, \dots, x_n, z] \cdot \Pi(z) \right)'' = \left(f[x_0, x_1, \dots, x_n, z]' \Pi(z) + f[x_0, x_1, \dots, x_n, z) \Pi'(z) \right)' =$$

$$= \underbrace{f[x_0, x_1, \dots, x_n, z]''}_{K=2} \Pi(z) + 2 \underbrace{f[x_0, x_1, \dots, x_n, z]'}_{K=1} \Pi'(x) + \underbrace{f[x_0, x_1, \dots, x_n, z]}_{K=0} \Pi''(z) =$$

$$= 2!\, f[x_0, x_1, \dots, x_n, z, \overline{z, z}] \cdot \Pi(z) + 2 \cdot f[x_0, x_1, \dots, x_n, z, \overline{z}] \Pi'(x) + f[x_0, x_1, \dots, x_n, z] \Pi''(z)$$

$$\hookrightarrow E = \frac{2 \cdot f^{\overset{\boxed{n+k+1}}{n+3)}}(\xi_1)}{(n+3)!} \cdot \Pi(z) + \frac{2 \cdot f^{n+2)}(\xi_2)}{(n+2)!} \Pi'(z) + \frac{f^{n+1)}(\xi_3) \Pi''(z)}{(n+1)!}$$

In our case: $x_0 = z-h$, $x_1 = z$, $x_2 = z+h$ $\rightarrow$ $\boxed{n=2}$

$$\begin{cases} \Pi(x) = (x-x_0)(x-x_1)(x-x_2) = (x-z+h)(x-z)(x-z-h) \\ \Pi'(x) = (x-z)(x-z-h) + (x-z+h)(x-z-h) + (x-z+h)(x-z) \\ \Pi''(x) = 2(x-z+h) + 2(x-z-h) + 2(x-z) \end{cases}$$

$\hookrightarrow$ z is a node : $x = z$

$$\begin{cases} \Pi(z) = (z-z+h)\cancelto{0}{(z-z)}(z-z-h) = 0 \\ \Pi'(z) = \cancelto{0}{(z-z)(z\dots)} + \underset{\boxed{h}}{(z-z+h)}\underset{\boxed{-h}}{(z-z-h)} + \cancelto{0}{(z\dots)(z-z)} = -h^2 \\ \Pi''(z)\ \ 2\underset{\boxed{h}}{(z-z+h)} + 2\underset{\boxed{-h}}{(z-z-h)} + \cancelto{0}{2(z-z)} = 0 \end{cases}$$

$$E = \cancelto{0}{\frac{2 \cdot f^{n+3)}(\xi) \cdot 0}{(n+3)!}} + \frac{2 f^{n+2)}(\xi_2)}{(n+2)!} \cdot (-h^2) + \cancelto{0}{\frac{f^{n+1)}(\xi_3) \cdot 0}{(n+1)!}}$$

$$\hookrightarrow (n=2) \rightarrow \boxed{E = -\frac{2 \cdot f^{4}(\xi)}{4!} \cdot h^2}$$

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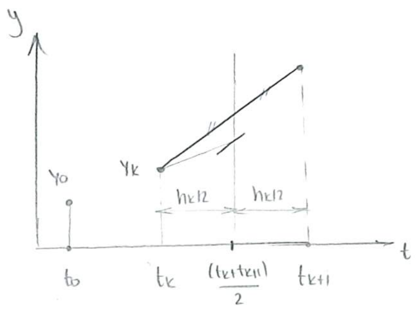
T 8. Write the formula of the Modified Euler (Midpoint) method to solve an ODE with initial conditions. Method: implicit/explicit? computational cost? OC? Family of methods it belongs?

MODIFIED EULER METHOD : EXPLICIT → y_{k+1} only appears as an output



$$\boxed{y_{k+1} = y_k + f(t_k + h_k/2,\ y_k + f(t_k, y_k)\cdot h_k/2)\cdot h_k}$$

$$\left.\begin{cases} k_1 = f(t_k, y_k)\cdot h_k \\ k_2 = f(t_k + h_k/2,\ y_k + k_1/2)\cdot h_k \end{cases}\right\} \boxed{y_{k+1} = y_k + k_2}$$

RUNGE KUTTA STYLE

- COMPUTATIONAL COST: 2 evaluations of $f(t,y)$ per step
- OC = 2
- As we have seen its family is: Runge-kutta → RK2

b) $\begin{cases} y_1' = -10y_1 - 20e^{t}y_2 \\ y_2' = 15e^{-t}y_1 - 10y_2 \end{cases}$ $\begin{cases} y_1(0) = 1 \\ y_2(0) = 2 \end{cases}$ $t \in [0,10]$ + TABLE

If $h = 0'1$ → Take one step of the method → cell C1

$$\left.\begin{cases} k_{11} = y_1'(0, [1,2])\cdot 0'1 = (-10-40)\cdot 0'1 = -5 \\ k_{12} = y_2'(0, [1,2])\cdot 0'1 = (15-20)0'1 = -0'5 \end{cases}\right\} \boxed{k_1 = (-5, -0'5)}$$

$$\left.\begin{cases} k_{21} = y_1'(0'05, [-1'5, 1'75])\cdot 0'1 = (15 - 36'794)0'1 = -2'179449 \\ k_{22} = y_2'(0'05, [-1'5, 1'75])\cdot 0'1 = (-21'4066 - 17'5)0'1 = -3'890266 \end{cases}\right\} \boxed{k_2 = (-2'179449, -3'890266)}$$

$$\boxed{C_1} = (1, 2) + (-2'179449, -3'890266) = \boxed{(-1'179449, -1'890266)}$$

c) knowing that the exact values at t=10 are $\begin{cases} y_1(10) = -8'2367\cdot 10^{-43} = -8236'7\cdot 10^{-46} \\ y_2(10) = -5'5013\cdot 10^{-46} = -5'5013\cdot 10^{-46} \end{cases}$

estimate C_2

THE METHOD IS OF ORDER 2 → p=2

↳ STEPS : 50 000 — 100000 → TOO MANY

$$\left(\frac{50\,000}{100\,000}\right)^p = \left(\frac{1}{2}\right)^2 = 1/4 = 0'25$$

TABLE

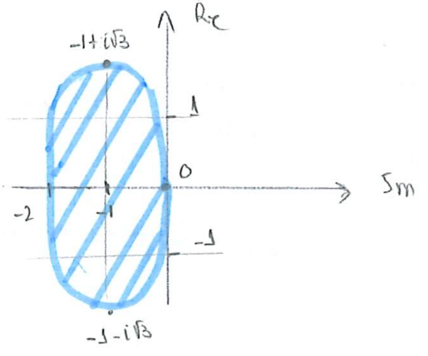
- $$\boxed{\frac{y_1(10) - y_{1,n,100000}}{y_1(10) - y_{1,n,50000}} \approx \frac{1}{4}} \longrightarrow 4(-8'2367\,10^{-43} + 8'2356\,10^{-43}) = -8'2367\,10^{-43} - y_{1,n,50000}$$

↳ $\boxed{y_{1,n,50000} = -8'2323\cdot 10^{-43}}$

- $$\frac{y_2(10) - y_{2,n,100000}}{y_2(10) - y_{2,n,50000}} \simeq \frac{1}{4} \longrightarrow 4(-5'5013 + 5'5006)\cdot 10^{-46} = -5'5013\cdot 10^{-46} - y_{2,n,500}$$

↳ $\boxed{y_{2,n,500} = -5'49985\,10^{-46}}$

$$\boxed{C_2 \simeq (-8'2323\cdot 10^{-43}, -5'4985\cdot 10^{-46})}$$



d). Absolute stability region :

- calculate the threshold step size from which the method will be unstable for that problem.

* ~~We define~~ J : $J = \begin{array}{c|cc|c} & y_1 & y_2 & \\ & -10 & -20e^t & y'_1 \\ & 15e^{-t} & -10 & y'_2 \end{array}$

↳ we are now calculating its eigenvalues.

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8. d) $|J-\lambda I|=0$

$$\begin{vmatrix} -10-\lambda & -20e^{t} \\ 15e^{-t} & -10-\lambda \end{vmatrix} = (10+\lambda)^2 + 300 = 0$$

$$100 + 20\lambda + \lambda^2 + 300 = 0 \rightarrow \lambda^2 + 20\lambda + 400 = 0$$

$$\lambda = \frac{-20 \pm \sqrt{400 - 4\cdot 400}}{2} = \frac{-20 \pm \sqrt{-3\cdot 20^2}}{2} = -10 \pm 10\sqrt{-3} = -10 \pm 10\sqrt{3}\,i$$

$$\text{eigs}(Jh) = h \cdot \text{eigs}(J)$$

$$-1 \pm \sqrt{3}i = h(-10 \pm 10\sqrt{3}i) \longrightarrow \boxed{h = 0'1}$$

e) what would happen using exact arithmetic

It would be the same. Those values come because the method having exponentials not because rounding errors.

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1. T

a) Define the concept of truncation error in one sentence

$$\overset{\text{exact.- approx}}{}$$

Is the error is the error incurred by a method when applying it with exact arithmetic.

b) From the form in div. differences of a Newton polynomial of degree 'n' at nodes: $x_0, x_1, \dots x_n$ prove that its truncation error at z satisfies:

$$e(z) = f[x_0, x_1, \dots, x_n, z]\,\pi(z) \quad \text{where} \quad \pi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

- $P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \dots + f[x_0,x_1,x_2,\dots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1}) =$

$$= \sum_{i=0}^{n} f[x_0,x_1,\dots,x_i](x-x_0)(x-x_1)\cdot(\dots)\cdot(x-x_{i-1})$$

If we add the node z:

- $P_{n+1}(x) = \sum_{i=0}^{n+1} f[x_0,\dots,x_i](x-x_0)(x-x_i)(\dots)(x-x_{i-1})$

- $\underbrace{P_{n+1}(z)}_{f(z)} = \underbrace{P_n(z)}_{P_n(z)} + \underbrace{f[x_0,\dots,x_n,z](x-x_0)(x-x_1)(\dots)(x-x_{n-1})(x-x_n)}_{e(z)}$

$$\hookrightarrow \boxed{e(z) = P_n(z) - f(z) = f[x_0,x_1,\dots,x_n,z](x-x_0)(x-x_1)\dots(x-x_{n-1})(x-x_n)}$$

c) From the previous expression, justify why when the nodes are equally spaced and $f(x_{n+1})$ is known at the new node x_{n+1}, the error:

$$e(x) \simeq \frac{\Delta^{n+1} f(x_0)}{(n+1)!}\, t(t-1)(t-2)\dots(t-n) \quad \text{with} \quad x = x_0 + th,\ h = x_{i+1} - x_i$$

$$f[x_0,x_1,\dots,x_n,x_{n+1}] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\downarrow$$

$$f[x_0, x_0+h, x_0+2h, \dots, x_0+nh, x_0+(n+1)h] = \frac{\Delta^{(n+1)} f(x_0)}{(n+1)!\, h^{n+1}}$$

$$e(x) = \frac{\Delta^{n+1} f(x_0)}{(n+1)!\, h^{n+1}} \cdot (x-x_0)(x-x_1)(x-x_2)\dots$$

CHANGE OF VARIABLE

$$\downarrow\ e(x) = \frac{\Delta^{n+1} f(x_0)}{(n+1)!\, \cancel{h^{n+1}}}\ t\cancel{h}\,(\cancel{h}(t-1))(\cancel{h}(t-2))\dots(\cancel{h}(t-n))$$

$$\hookrightarrow e(x) = \frac{\Delta^{n+1} f(x_0)}{(n+1)!}\ t(t-1)(t-2)\dots(t-n)$$

2.

x_i	0'5	0'55	0'6	0'65	0'7
$f(x_i)$	0'7788	0'73897	0'69768	0'65541	0'61263

a) obtain an approx. value of x for f(x) - 0'7 by means a polymomial of degree 3 evaluated optimally

$f_i(x)$	$f^{-1}(x_0)$	$f^{-1}(x_i)$	$f^{-1}(x_{i,1})$	$f^{-1}(x_{i,2})$	$f^{-1}(x_{i,4})$
0'7788	0'5	-	-	-	-
0'73897	0'55	-1'2553517	-	-	-
0'69768	0'6	-1'210947	-0'54719	-	-
0'65541	0'65	-1'182872	-0'335986	-1'711678	-
0'61263	0'7	-1'16877	-0'165808	-1'32354	-2'13357886 5

The first 4 (for degree = 3) are the closest to f(x) = 0'7

$$x = q_3 = 0'5 - 1'2553517\,(Y-0'7788) - 0'54719(Y-0'7788)(Y-0'73897) - 1'711678\,(Y-0'7788)(Y-0'73897)(Y-0'69768)$$

$$x = 0'5 + \underbrace{(Y-0'7788)\underbrace{\left[-1'2553517 + \underbrace{(Y-0'73897)\underbrace{\left[-0'54719 - \underbrace{1'711678\underbrace{(Y-0'69768)}_{0'00232}}_{0'0039711}\right]}_{-0'5511611}}_{0'0214987747}\right]}_{-1'233872952}}_{0'097229188}$$

$$y = 0'7$$

$$\hookrightarrow \boxed{x = 0'597229}$$

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2. b) Estimate the error made

$$q_4 = 0'5 - 1'255317(y-0'7788) - 0'54719(y-0'7788)(y-0'7399) - 1'711678(y-0'7788)(y-0'73897)(y-0'69768)$$

$$- 2'33578865(y-0'778)(y-0'7397)(y-0'69768)(y-0'65541)$$

$$\boxed{e(0'7) = q_4 - q_3} = -2'33578865\underbrace{(0'7-0'7788)}_{-0'0788}\underbrace{(0'7-0'7397)}_{-0'0397}\underbrace{(0'7-0'69768)}_{0'00232}\underbrace{(0'7-0'65541)}_{0'04459} = \boxed{-7'56\cdot 10^{-6}}$$

3. Calculate the values of f at $-1, -0'5, 0, 0'5, 1$ knowing that the midpoint rule to approx: $\int_{-1}^{1} f(x)\,dx$ yields 12 and the compound midpoint rule yields 5 and the compound simpson rule yields 6.

$f(-1) = f(1)$

$f(-0'5) = f(0'5) - 1$

x_i	-1	$-0'5$	0	$0'5$	1
$f(x_i)$	f_0	f_1	f_2	f_3	f_4

- **MIDPOINT RULE**: $2\cdot \underset{\llcorner=1}{h}\cdot f_2 = 12 \longrightarrow \boxed{f_2 = f(0) = \frac{12}{2\cdot 1} = 6}$

- **COMPOUND MIDPOINT**: $\underset{\llcorner=1}{h}(f_1+f_3) = 5 \longrightarrow f_1+f_3 = 5$

- **COMPOUND SIMPSON**: $\frac{\overset{=0'5}{h}}{3}(f_0+4f_1+f_2) + \frac{h}{3}(f_2+4f_3+f_4) = \frac{h}{3}(f_0+4f_1+2f_2+4f_3+f_4)$

$\llcorner \frac{0'5}{3}(f_0+4f_1+2\cdot 6+4f_3+f_4) = 6 \longrightarrow 36 = f_0+4(f_1+f_3)+12+f_4$

$f_0 + 4\cdot 5 + 12 + f_4 = 36 \longrightarrow f_0+f_4 = 4$

$\rangle\ f_0 = f_4 = 2 \longrightarrow \boxed{f(-1) = f(1) = 2}$

- $f(-1) = f_0 = f(1) = f_4 \longrightarrow f_0 = f_4$

- $f(-0'5) = f(0'5) - 1 \longrightarrow f_1 = f_3 - 1$

$\longrightarrow \boxed{f(0'5) = 3}$

$\llcorner 2f_3 = 6 \longrightarrow f_3 = 3 \quad \llcorner \boxed{f(-0'5) = 2}$

4. Laguerre polynomials verify:

$$L_{n+1}(x) = (2n+1-x)\,L_n(x) - n^2 L_{n-1}(x) \quad \text{with} \quad L_0(x) = 1, \; L_1(x) = 1-x$$

a) Derive the quadrature rule to approx. the integrals of the form $I = \int_0^{\infty} e^{-x} f(x)\,dx$ using the roots of $L_2(x)$. What is the polynomial degree?

$$I = \int_0^{\infty} e^{-x} f(x)\,dx$$

$$L_2(x) = (2+1-x)\,L_1(x) - 1^2 L_0(x)$$

$$L_2(x) = (3-x)(1-x) - 1 = 3 - 4x + x^2 - 1 = x^2 - 4x + 2$$

$\hookrightarrow$ roots or $L_2(x)$:

$$x^2 - 4x + 2 = 0 \longrightarrow x = \frac{+4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2} \begin{matrix} \nearrow \boxed{x_1 = 2+\sqrt{2}} \\ \searrow \boxed{x_2 = 2-\sqrt{2}} \end{matrix}$$

- $I = \int_0^{\infty} e^{-x} f(x)\,dx = W_1 \cdot f(2+\sqrt{2}) + W_2\, f(2-\sqrt{2})$

- $f(x) = 1 \rightarrow \int_0^{\infty} e^{-x} \cdot 1\,dx = -e^{-x}\Big]_0^{\infty} = 1 = W_1 + W_2 \longrightarrow \boxed{W_1 + W_2 = 1}$

- $f(x) = x \longrightarrow \int_0^{\infty} e^{-x} \cdot x\,dx = \left[x(-e^{-x})\Big|_0^{\infty} + \int_0^{\infty} e^{-x}\,dx\right] = \left[-\cancelto{\emptyset}{xe^{-x}} - e^{-x}\right]_0^{\infty} = \boxed{1 = W_1(2+\sqrt{2}) + W_2(2-\sqrt{2})}$

$$\uparrow \left\{ \begin{matrix} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x} \end{matrix} \right\}$$

$$\left\{ \begin{array}{l} W_1 = 1 - W_2 \\ \\ 1 = (1-W_2)(2+\sqrt{2}) + W_2(2-\sqrt{2}) \\ \hookrightarrow 1 = 2+\sqrt{2} - 2\sqrt{2}W_2 \rightarrow -2\sqrt{2}W_2 = -\sqrt{2} - 1 \rightarrow \boxed{W_2 = \frac{\sqrt{2}+1}{2\sqrt{2}} = \frac{2+\sqrt{2}}{4}} \end{array} \right.$$

$$\hookrightarrow \boxed{W_1 = 1 - \frac{2+\sqrt{2}}{4} = \frac{2-\sqrt{2}}{4}}$$

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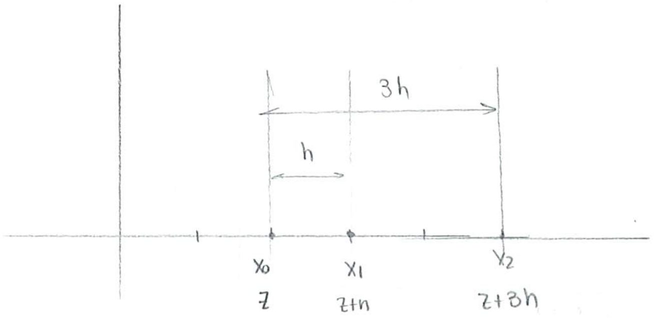
4.
$$\boxed{I = \int_0^{\infty} e^{-x} f(x)\,dx = \frac{(2-\sqrt{2})}{4} \cdot f(2+\sqrt{2}) + \frac{(2+\sqrt{2})}{4} f(2-\sqrt{2})}$$

POLYNOMIAL DEGREE : $N = 2n+1 = \boxed{3}$

5. T we want to estimate $f'(z)$ using: $x_0 = z$; $x_1 = z+h$ $x_2 = z+3h$

a) Obtain the diff formulas using Lagrange basic pol.



$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

SUBSITUTING $\begin{cases} x_0 = z \\ x_1 = z+h \\ x_2 = z+3h \end{cases}$

$$P_2(x) = \frac{(x-z-h)(x-z-3h)}{(-h)(-3h)} f(x_0) + \frac{(x-z)(x-z-3h)}{(h)(-2h)} f(x_1) + \frac{(x-z)(x-z-h)}{(3h)(2h)} f(x_2)$$

$$P_2(x) = \underbrace{\frac{(x-z-h)(x-z-3h)}{3h^2}}_{L_0(x)} f(x_0) + \underbrace{\frac{(x-z)(x-z-3h)}{-2h^2}}_{L_1(x)} f(x_1) + \underbrace{\frac{(x-2)(x-z-h)}{6h^2}}_{L_2(x)} f(x_2)$$

$$\cdot\ L_0'(x) = \frac{(x-z-3h)+(x-z-h)}{3h^2} = \frac{2x-2z-4h}{3h^2}$$

$$\cdot\ L_1'(x) = \frac{(x-z-3h)+(x-z)}{-2h^2} = \frac{2x-2z-3h}{-2h^2}$$

$$\cdot\ L_2'(x) = \frac{(x-z)+(x-z-h)}{6h^2} = \frac{2x-2z-h}{6h^2}$$

↳ As z is a node: $x=z$

$$L_0'(z) = \frac{\cancel{2z}-\cancel{2z}-4h}{3h^2} = \frac{-4}{3h} = A_0$$

$$L_1'(z) = \frac{\cancel{2z}-\cancel{2z}-3h}{-2h^2} = \frac{3}{2h} = A_1$$

$$L_2'(z) = \frac{2z-\cancel{2z}-h}{6h^2} = -\frac{1}{6h} = A_2$$

$$\boxed{f'(z) = A_0 \cdot f(x_0) + A_1 f(x_1) + A_2 f(x_2) + E}$$

$$\boxed{f'(z) = \frac{-4}{3h} \cdot f(z) + \frac{3}{2h} f(z+h) - \frac{1}{6h} f(z+3h) + E = \frac{-8f(z)+9f(z+h)-f(z+3h)}{6h} + E}$$

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b) Obtain its truncation error from the interp. error.

$$E = e'(z) = f'[x_0, x_1, x_2, z] \cdot \Pi(z) + f[x_0, x_1, x_2, z]\ \Pi'(z)$$

$$\circ\ \Pi(x) = (x-x_0)(x-x_1)(x-x_2) = (x-z)(x-z-h)(x-z-3h)$$

$$\circ\ \Pi'(x) = (x-x_0)(x-x_1) + (x-x_0)(x-x_2) + (x-x_1)(x-x_2) = (x-z)(x-z-h) + (x-z)(x-z-3h) + (x-z-h)(x-z-3h)$$

$x=z$

$$\Pi(z) = \overset{0}{\cancel{(z-z)}}(\cdots)(\cdots) = 0$$

$$\Pi'(z) = 0 + 0 + (-h)(-3h) = 3h^2$$

$$E = f[x_0, x_1, x_2, z] \cdot 3h^2 = \frac{f^{2+1)}(\xi)}{(2+1)!} 3h^2 \longrightarrow \boxed{E = \frac{f^{3)}(\xi)\, h^2}{2}}$$

$\hookrightarrow 3\cdot 2\cdot 1$

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5.T c) obtain h_{opt} in terms of the upper bound ε of the abs. value of the error in the values of f and the upper bound M of the abs. value of the derivative of f that appears in the error term.

$$f'(z) = \underbrace{\frac{-8f(z) + 9f(z+h) - f(z-3h)}{6h}}_{D} + \underbrace{\frac{f^{3}(\xi)h^{2}}{2}}_{E}$$

STEPS :

1. $|E| \leq \bigcirc \cdot M = g_1(h)$
2. $|AF| = \sum_{i=0}^{n} |A_i|$
3. $|E_r| = \varepsilon |AF| = g_2(h)$
4. $g(h) = g_1(h) + g_2(h)$
5. $g'(h) = 0$

1\. $\cdot\ |E| = \left| \frac{f^{3)}(\xi) h^2}{2} \right| \leq \frac{h^2}{2} \cdot M = g_1(h)$

2\. $\cdot\ |AF| = \sum_{i=0} |A_i| = \left|\frac{-4}{3h}\right| + \left|\frac{3}{2h}\right| + \left|-\frac{1}{6h}\right| = \frac{8+9+1}{6h} = \frac{18}{6h} = \frac{3}{h}$

3\. $\cdot\ |E_r| = \varepsilon |AF| = \frac{3\varepsilon}{h} = g_2(h)$

4\. $\cdot\ \boxed{g(h) = g_1(h) + g_2(h)} = \frac{h^2 M}{2} + \frac{3\varepsilon}{h} = \frac{h^2 M}{2} + 3h^{-1}\varepsilon$

5\. $g'(h) = \frac{\cancel{2}hM}{\cancel{2}} - 3h^{-2}\varepsilon = 0 \rightarrow hM = \frac{3}{h^2}\varepsilon \rightarrow h^3 = \frac{3\varepsilon}{M} \rightarrow \boxed{h_{opt} = \sqrt[3]{\frac{3\varepsilon}{M}}}$

6\. 2 consistent linear multistep methods have been combined to create → P(EC)E (predictor corrector)

$P_1(z) = 2z$; $P_2(z) = (5z^2 + 8z - 1)/12$; $P_3(z) = z^2 - z$; $P_4(z) = z^2 - 1$

a) Justify what the formulas of the predictor and the corrector methods are

* A METHOD IS CONSISTENT if $\boxed{\rho(1) = 0}$ and $\boxed{\rho'(1) = \sigma(1)}$

- $P_1(1) = 2 \neq 0 \longrightarrow \boxed{P_1(1) = 2}$
- $P_2(1) = (5 + 8 - 1)/12 = 1 \neq 0 \longrightarrow P_2(1) = 1$
- $\boxed{P_3(1) = 1 - 1 = 0} \longrightarrow p'_3(1) = 2 - 1 = 1$
- $\boxed{P_4(1) = 1 - 1 = 0} \longrightarrow \boxed{p'_4(1) = 2}$

$\boxed{\rho(z) = P_4,\ \sigma(z) = P_1}$ ①

$\boxed{\rho(z) = P_3,\ \sigma(z) = P_2}$ ②

$$* \quad \boxed{\rho(z) = \sum_{j=0}^{k} \alpha_j \cdot z^j \quad ; \quad \sigma(z) = \sum_{j=0}^{k} \beta_j z^j}$$

$$① \begin{cases} \rho_1(z) = z^2 - 1 \longrightarrow \boxed{\alpha_0 = -1, \ \alpha_1 = 0, \ \alpha_2 = 1} \\ \sigma_1(z) = 2z \longrightarrow \boxed{\beta_0 = 0, \ \beta_1 : 2, \ \beta_2 = 0} \end{cases} \quad (k = 2)$$

$$② \begin{cases} \rho_2(z) = z^2 - z \longrightarrow \boxed{\alpha_0 = 0, \ \alpha_1 = -1, \ \alpha_2 = 1} \\ \sigma_2(z) = \frac{5}{12} z^2 + \frac{8}{12} z - \frac{1}{12} \longrightarrow \boxed{\beta_0 = -1/12, \ \beta_1 : 8/12, \ \beta_2 = 5/12} \end{cases} \quad (k = 2)$$

$$\boxed{y_{n+k} = -\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h \sum_{j=0}^{k} \beta_j f_{n+j}}$$

$$① \cdot \ y_{n+2} = -(-1 \cdot y_n + 0 \cdot y_{n+1}) + (0 \cdot f_n + 2 \cdot f_{n+1} + 0 \cdot f_{n+2})h \longrightarrow \boxed{y_{n+2} = y_n + 2h \cdot f_{n+1}} \quad (P_4, P_1)$$

$$② \cdot \ y_{n+2} = -(0 \cdot y_n - 1 \, y_{n+1}) + \left(-\frac{1}{12} f_n + \frac{8}{12} f_{n+1} + \frac{5}{12} f_{n+2}\right)h \rightarrow \boxed{y_{n+2} = y_{n+1} + \frac{h}{12}\left(-f_n + 8 f_{n+1} + \underbrace{5 f_{n+2}}_{\text{IMPLICIT}}\right)} \quad (P_2, P_3)$$

b) Analyze the stability and the OC. of both methods

* STABLE if : |roots $\rho(z)$| ≤ 1 and if $1 \equiv$ simple

$$① \ \rho_1(z) : z^2 - 1 \longrightarrow z^2 - 1 = 0 \begin{array}{l} \nearrow z = -1 \\ \searrow z = 1 \ \text{(SIMPLE)} \end{array} \longrightarrow |z| \leq 1 \longrightarrow \text{STABLE}$$

$$② \ \rho_2(z) = z^2 - z \longrightarrow z(z-1) = 0 \begin{array}{l} \nearrow z = 0 \\ \searrow z = 1 \ \text{(SIMPLE)} \end{array} \longrightarrow |z| \leq 1 \longrightarrow \text{STABLE}$$

* CONVERGENCE OF ORDER P : $\boxed{\frac{1}{m} \sum_{j=0}^{k} \alpha_j \cdot j^m = \sum_{j=0}^{k} \beta_j \, j^{m-1}}$

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6.

①

b)

- $m=1 \longrightarrow \underbrace{-1\cdot 0^1 + 0\cdot 1^1 + 1\cdot 2^1}_{2} = \underbrace{0\cdot 0^0 + 2\cdot 1^0 + 0\cdot 2^0}_{2}$ (2=2) ✓

- $m=2 \longrightarrow \frac{1}{\cancel{2}}(1\cdot 2^{\cancel{2}}) = 2\cdot 1^1$ (2=2) ✓

- $m=3 \longrightarrow \frac{1}{3}(1\cdot 2^3) = 2\cdot 1^2$ (8/3 ≠ 2) x $\longrightarrow \boxed{p=2}$

② . $m=1 \longrightarrow \underbrace{0\cdot 0^1 - 1\cdot 1^1 + 1\cdot 2^1}_{1} = \underbrace{-\frac{1}{12}0^0 + \frac{8}{12}1^0 + \frac{5}{12}2^0}_{1}$ (1=1) ✓

- $m_2 \longrightarrow \frac{1}{2}(\underbrace{-1\cdot 1^2 + 1\cdot 2^2}_{3}) = \frac{1}{12}(\underbrace{8\cdot 1 + 5\cdot 2}_{18})$ (1'5 = 1'5) ✓

- $m_3 \longrightarrow \frac{1}{3}(-1+8) = \frac{1}{12}(8+20)$ (7/3 = 28/12) ✓

- $m_4 \longrightarrow \frac{1}{4}(-1+16) = \frac{1}{12}(8+40)$ (15/4 ≠ 4) $\longrightarrow \boxed{p=3}$

c) OC for the predictor - corrector method

The predictor - corrector method is of order $\boxed{p=3}$

d) compare its computational cost with that of Raunge kutta of the same order

- RK3 → 3 evaluations per step
- PREDICTOR - CORRECTOR → 2 evaluations per step

> $\boxed{\text{COMP. COST (RK3)} = \frac{3}{2}\ \text{COMP. COST (PRED-CORR. m)}}$

e) $$\begin{cases} h = 0'05 \\ y''(t) = g(t) + g(t^2) \cdot y(t) \qquad t \in [0, 10] \\ y(0) = y'(0) = 666 \end{cases}$$

- where : $g(t)$ is a known function whose eval.'s comp. cost $= 3 \cdot 10^3$ floating-point op.
- Computer: 10^6 operations per second

Approx time ?

$$\frac{10-0}{n} = 0'05 \longrightarrow n = 200 \text{ steps}$$

$$\left[t = 200 \text{ } \cancel{\text{steps}} \cdot \frac{2 \cancel{\text{eval } f}}{\cancel{\text{step}}} \cdot \frac{2 \cancel{\text{eval} g}}{1 \cancel{\text{eval} f}} \cdot \frac{3 \cdot 10^3 \cancel{\text{op}}}{1 \cancel{\text{eval} g}} \cdot \frac{1 s}{10^6 \cancel{\text{op}}} = \boxed{2'4 s} \right.$$

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1. Define the concept of an osculating polynomial of a function f(x) for the nodes: $x_0, x_1, \dots, x_n \in [a,b]$ and write an expression of its truncation error when k interp. data are used

- An osculating polynomial p(x) of f(x) is one that must coincide with f(x) on its values and number of successive derivates of the nodes:

$$\boxed{P(x_i) = f(x_i) \; ; \; p'(x_i) = f'(x_i) \; ; \; P''(x_i) = f''(x_i) \; ; \dots , \; P^{m_i)}(x_i) = f^{m_i)}(x_i) \qquad (i = 0, 1, \dots, n)}$$

- Number of conditions at x_i : $\boxed{k_i = m_i + 1}$ ⟶ Number of conditions for the polynomial : $\boxed{K = \sum_{i=0}^{n} k_i}$

- Assuming $f \in C^k$:

$$\boxed{e(x) = f(x) - p(x) = \frac{f^{k)}(\xi)}{k!} (x-x_0)^{k_0} (x-x_1)^{k_1} \cdots (x-x_n)^{k_n}}$$

2

x_i	0	1	3	4	5	6
$f(x_i)$	1	6	70	153	86	481

Knowing that $f'''(x) = 12 \quad \forall x \in \mathbb{R}$, correct the wrong datum of the table of diff

x_i	$f_{i,0}$	$f_{i,1}$	$f_{i,2}$	$f_{i,3}$	$f_{i,4}$	$f_{i,5}$
0	1	—	—	—	—	—
1	6	5	—	—	—	—
3	70	32	9	—	—	—
4	153	83	17	2	—	—
5	286	133	25	2	0	—
6	481	195	31	2	0	0

* INTUITIVELY $f(x_4) = 86$ will be WRONG

$2 = \frac{f_{42} - 17}{4} \rightarrow 25$

$25 = \frac{f_{41} - 83}{2} \rightarrow 133$

$133 = \frac{f_{40} - 153}{1} \rightarrow \boxed{f(x_4) = 286}$

$f_{51} = 195$

$$\begin{cases} f^{3)}(x) = 12 \\ \dfrac{f^{3)}(x)}{3!} = f_{i,3} = \dfrac{12}{3\cdot 2} = 2 \end{cases}$$ → as $f^{3)}(x) \equiv kte$ $\forall x$ → the following columns $= 0$.

b) Exclusively using the interpolation points corresponding to $x_i = 1, 3, 4$ and without calculating $f(x)$, obtain $f'(3)$

* we are building our table as if we had an osculating pol.

x_i	f_{0i}	f_{1i}	f_{2i}	f_{3i}
1	6	—	—	—
3	70	32	—	—
3	70	$f'(3)/1!$	a	—
4	153	83	b	2

$$\left.\begin{array}{l} \dfrac{b-a}{3} = 2 \longrightarrow b-a = 6 \\ \dfrac{f'(3)-32}{2} = a \\ \dfrac{83-f'(3)}{1} = b \end{array}\right\} \quad 83 - f'(3) - \frac{f'(3)+32}{2} = 6 \longrightarrow 186 = 3f'(3) \longrightarrow \boxed{f'(3) = 62}$$

c) obtain $f(x)$ usin finite diff and evaluate optimally at $x = 3'5$ $(h=1)$

	Δ_{0i}	Δ_{1i}	Δ_{2i}	Δ_{3i}
3	70	—	—	—
4	153	83	—	—
5	286	133	50	—
6	481	195	62	12

* CHANGE OF VARIABLE :

$\boxed{x = x_0 + ht}$

$x = 3 + t \longrightarrow t = x - 3 \longrightarrow t = 0'5$

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2. c)

$$q(t) = 70 + 83t + 50\frac{(t-1)t}{2!} + 12\frac{(t-1)(t-2)t}{3!}$$

$$q(t) = \underbrace{70 + \underbrace{t\left[\underbrace{83 + \underbrace{(t-1)\left(\underbrace{25 + \underbrace{2(t-2)}_{-3}}_{22}\right)}_{-11}}_{72}\right]}_{36}}_{106} \qquad \boxed{t=0'5}$$

$$\boxed{f_x(3'5) = q_t(0'5) = 106}$$

3. $\displaystyle\int_0^{1/2} \frac{e^{-4x^2}}{\sqrt{1-4x^2}}\,dx$ with a precision of $\underbrace{1\%}_{\varepsilon = 0'01}$ using Gauss quadrature

1. CHANGE OF VARIABLE :

- $\boxed{4x^2 = t^2}$

  ↳ $x = t/2 \longrightarrow \boxed{dx = dt/2}$

- Integration limits : $t = 2x$ $\begin{cases} t_1 = 2\cdot 0 = 0 \\ t_2 = 2\cdot\frac{1}{2} = 1 \end{cases}$

$$Q = \int_0^{1/2} \frac{e^{-4x^2}}{\sqrt{1-4x^2}}\,dx = \int_0^1 \frac{e^{-t^2}}{2\sqrt{1-t^2}}\,dt = \frac{1}{2}\int_{-1}^{1} \frac{e^{-t^2}}{2\sqrt{1-t^2}}\,dt$$

$$\hookrightarrow Q = \int_{-1}^{1} \frac{e^{-t^2}/4}{\sqrt{1-t^2}}\,dt \longrightarrow \text{GAUSS CHEBYSHEV}$$

$$t_i = \cos\left(\frac{\pi/2 + i\pi}{n+1}\right)$$

$$w_i = \frac{\pi}{n+1}$$

$$f(t) = e^{-t^2}$$

$$w = \pi$$

- $n=0$

$$\begin{cases} t_0 = \cos(\pi/2) = 0 \\ w_0 = \pi \end{cases} \longrightarrow \boxed{Q_1 = \frac{\pi}{4}\cdot f(0) = \frac{\pi}{4} = 0'785398163}$$

- $n=1$

$$\begin{cases} t_0 = \sqrt{2}/2 \\ t_1 = -\sqrt{2}/2 \\ w = \pi/2 \end{cases} \longrightarrow \boxed{Q_2 = \frac{\pi}{8}\left[e^{-0'5} + e^{-0'5}\right] = 0'473675}$$

$$\hookrightarrow \left(\varepsilon = |Q_2 - Q_1| = 0'311723163\right)$$

- $n=2$

$$\begin{cases} t_0 = \sqrt{3}/2 \\ t_1 = 0 \\ t_2 = -\sqrt{3}/2 \\ w = \pi/3 \end{cases} \longrightarrow \boxed{Q_3 = \frac{\pi}{12}\left[2\cdot e^{-0'75} + 1\right] = 0'509129936}$$

$$\hookrightarrow \left(\varepsilon = |Q_3 - Q_2| = 0'035455 > 0'01\right.$$

- $n=3$

$$\begin{cases} t_0 = 0'923879532 \\ t_1 = 0'382683432 \\ t_2 = -t_1 \\ t_3 = -t_0 \\ w = \pi/4 \end{cases} \longrightarrow \boxed{Q_4} = \frac{\pi}{16}\left[2\cdot e^{-0'8535534} + 2\cdot e^{-0'4644661}\right] = \boxed{0'5064525}$$

$$\hookrightarrow \underline{\varepsilon =} |Q_4 - Q_3| = \underline{0'002677 < 0'01}$$

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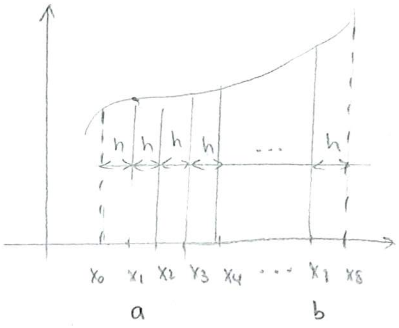
T4 consider the Newton - Cotes formula: $\int_{x_0}^{x_8} f(x)\,dx = \sum_{i=1}^{7} A_i f(x_i) + K h^r \cdot f^{m)}(\xi)$ with $\xi \in [x_0, x_8]$

where $x_i = x_0 + ih$ for $i = 1, 2, \dots, 8$

a) say what the values of r and m are. Say if there are other Newton Cotes formulas with the same OC and say which one would you use and why.

$$\int_{x_0}^{x_8} f(x)\,dx = A_1 f(x_1) + A_2 f(x_2) + \dots + A_7 f(x_7) + K h^r f^{m)}(\xi)$$



↳ x_0, x_8 are not included → OPEN NEWTON COTE

- so our nodes will be: $x_1, x_2, x_3, x_4, x_5, x_6, x_7$

$\to$ nn = 7 (ODD)

$\to$ min. pol. degree = 6

$> N = 6 + 1 = 7$

$m = N + 1 \longrightarrow \boxed{m = 8}$

$r = m + 1 \longrightarrow \boxed{r = 9} \longrightarrow OCS \equiv r = \underline{9} = N + 2$

*NEWTON COTES WITH SAME OC:

- CNC of 7 nodes
- CNC of 8 nodes
- ONC of 8 nodes

CLOSED NC ARE MORE STABLE

ODD nn → MORE "EFFICIENT" → LESS COMP. COST

↳ CNC of 7 NODES

4. $f(x) \in C^m([a,b])$, derive the expression of the trunc. error of the formula that results from composing the one of section a) M times in $[a,b]$

$$E_{CS} = \sum_{i=1}^{M} K \cdot h^9 \cdot f^{8)}(\xi) = k h^9 \cdot M \sum_{i=1}^{M} \frac{f^{8)}(\xi)}{M} = k h^9 \cdot M \cdot \overline{f}^{8)}(\xi)$$

Knowing that $M = \frac{(b-a)}{8h}$:

$$E_{CS} = \frac{k\, h^{\overset{8}{\cancel{9}}} \cdot (b-a)}{8 \cancel{h}} \cdot \overline{f}^{8)}(\xi) \rightarrow \boxed{E_{CS} = \frac{K(b-a)h^8}{8} \overline{f}^{8)}(\xi) \quad \text{f.s } \xi \in [a,b]}$$

5. $Y(t) = \sqrt{t}$

$$\left\{ y'' = -\frac{1}{4ty} \; ; \quad y(1) = 1 \; ; \quad y'(1) = 0'5 \right. \rightarrow f(t_k, y_k) = -\frac{1}{4tY}$$

Approx the value of $\sqrt{2}$ applying the Enhanced Euler method with $h = 0'5$.

ENHANCED EULER METHOD :

$$\boxed{Y_{K+1} = Y_K + \frac{f(t_k, Y_k) + f(t_k + h_k, Y_k + f(t_k, Y_k) h_k)}{2} \cdot h_k}$$

$$\begin{cases} K_1 = f(t_k, Y_k)\, h_k \\ K_2 = f(t_k + h_k, K_1 + Y_k)\, h_k \end{cases} \longrightarrow \boxed{Y_{K+1} = Y_K + \frac{K_1 + K_2}{2}}$$

- OUR GOAL : $Y = \sqrt{2} \rightarrow \boxed{t = 2} \rightarrow t_f = 2$

$$\frac{2-1}{k} = h = 0'5 \longrightarrow 2 \text{ STEPS}$$

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5.

$$\begin{cases} y = t^{1/2} \rightarrow y(1) = 1 \ \checkmark \\ y' = - t^{-1/2}/2 \rightarrow y'(1) = 0'5 \ \checkmark \\ y'' = -1/ty4 \ \checkmark \end{cases}$$

*** STABLISHING THE SYSTEM OF ODE'S**

$$\mathcal{Y}_k = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\mathcal{Y}'_k = \begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y_2 \\ -0'25/ty \end{pmatrix} \longrightarrow \begin{cases} \mathcal{Y}_k = \begin{pmatrix} y \\ y' \end{pmatrix} \\ f(t_k, y_k) = \dfrac{\partial y_k}{\partial y} = \begin{pmatrix} y_2 \\ -0'25/ty \end{pmatrix} \end{cases}$$

$\boxed{k=0}$ $t = 1$; $\mathcal{Y}_k = \begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0'5 \end{pmatrix}$

$$k_1 = 0'5 \begin{pmatrix} 0'5 \\ -0'25/1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0'25 \\ -0'125 \end{pmatrix}$$

$$k_2 = 0'5 \cdot f\left(1'5, \begin{pmatrix} 1'25 \\ +0'375 \end{pmatrix}\right) = 0'5 \begin{pmatrix} +0'375 \\ \dfrac{-0'25}{1'5 \cdot 1'25} \end{pmatrix} = \begin{pmatrix} +0'1875 \\ -1/15 \end{pmatrix}$$

$$\mathcal{Y}(1'5) = \begin{pmatrix} 1 \\ 0'5 \end{pmatrix} + \begin{pmatrix} 0'25 + 0'1875 \\ -0'125 - 1/15 \end{pmatrix} \cdot \frac{1}{2} = \begin{pmatrix} 1'21875 \\ 0'404167 \end{pmatrix}$$

$\boxed{K=1}$ $t=1'5$; $\underset{\sim}{Y_k} = Y(1'5) = \begin{pmatrix} 1'21875 \\ 0'404167 \end{pmatrix}$

$$k_1 = 0'5 \begin{pmatrix} 0'404167 \\ -0'25/1'5 \cdot 1'21875 \end{pmatrix} = \begin{pmatrix} 0'2020835 \\ -0'0683607 \end{pmatrix}$$

$$K_2 = f\left(2, \begin{pmatrix} 1'4208335 \\ 0'3358063 \end{pmatrix}\right) \cdot 0'5 = 0'5 \cdot \begin{pmatrix} 0'3358063 \\ 0'087977652 \end{pmatrix} = \begin{pmatrix} 0'16790315 \\ 0'043988264 \end{pmatrix}$$

$$\underset{\sim}{Y}(2) = \begin{pmatrix} 1'21875 \\ 0'404167 \end{pmatrix} + \begin{pmatrix} 0'36998665 \\ -0'02437243 6 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 1'403743325 \\ 0'391980782 \end{pmatrix}$$

$$\underset{\sim}{Y}(2) = \begin{pmatrix} Y(2) \\ Y'(2) \end{pmatrix} \longrightarrow \boxed{Y(2) = 1'403743325 \simeq \sqrt{2}}$$

T 6. Obtain a linear multistep method to solve init. value problems:

(P)

* CHARACTERISTICS :
- 2 steps ✓
- convergent ✓
- Explicit ✓
- The sum of the roots of each one of its 2 charac. pol. $=0$ ✓

Calculate the order of the method

- 2 STEPS $\longrightarrow \boxed{k=2}$ ✓

- Advanced formula: $Y_{n+2} = \sum_{j=0}^{1} \alpha_j Y_{n+j} + h \cdot \sum_{j=0}^{2} \beta_j f_{n+j}$

$$\hookrightarrow Y_{n+2} = \alpha_0 Y_n + \alpha_1 Y_{n+1} + h\left(\beta_0 f_n + \beta_1 f_{n+1} + \underbrace{\beta_2 f_{n+2}}_{\boxed{\beta_2=0} \to \text{EXPLICIT ✓}}\right)$$

$$\hookrightarrow \boxed{Y_{n+2} = \alpha_0 Y_n + \alpha_1 Y_{n+1} + h\left(\beta_0 f_n + \beta_1 f_{n+1}\right)}$$

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6.

* $\rho(z) = \sum_{i=0}^{2} \alpha_i z^i$; $\sigma(z) = \sum_{j}^{k} \beta_j z^j$

$$\boxed{\rho(z) = \alpha_0 + \alpha_1 z + z^2} \longrightarrow \boxed{\rho'(z) = \alpha_1 + 2z}$$

$$\boxed{\sigma(z) = \beta_0 + \beta_1 z}$$

- CONVERGENCY: $\rho(1) = 0$; $\rho'(1) = \sigma(1)$
(consistency)

$$\rho(1) = \alpha_0 + \alpha_1 + 1 = 0 \longrightarrow \boxed{\alpha_0 + \alpha_1 = -1}$$

$$\left.\begin{array}{l} \rho'(1) = \alpha_1 + 2 \\ \sigma(1) = \beta_0 + \beta_1 \end{array}\right\} \boxed{\beta_0 + \beta_1 = \alpha_1 + 2}$$

- ROOTS:

$$\alpha_0 + \alpha_1 z + z^2 = 0 \longrightarrow z = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_0}}{2} \longrightarrow z_1 + z_2 = 0 \longrightarrow \boxed{\alpha_1 = 0}$$

$$\beta_0 + \beta_1 z = 0 \longrightarrow z = -\beta_0 / \beta_1 \longrightarrow \boxed{\beta_0 = 0}$$

$$\begin{cases} \alpha_0 + 0 = -1 \longrightarrow \boxed{\alpha_0 = -1} \\ 0 + \beta_1 = 0 + 2 \longrightarrow \boxed{\beta_1 = 2} \end{cases}$$

$$\boxed{Y_{n+2} = -Y_n + 2h f_{n+1}}$$

• ORDER OF CONVERGENCE: $\frac{1}{m}\sum_{j=0}^{k}\alpha_j j^m = \sum_{j=0}^{k}\beta_j j^{m-1}$

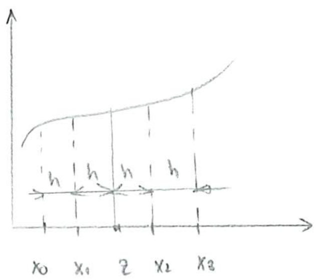
(m=1) $\underbrace{1\left(-1\cdot 0+1\cdot 2^1\right)}_{2} = \underbrace{2\cdot \cancel{1}}_{2}$ $0 \to 1$ ✓

(m=2) $\underbrace{\frac{1}{2}\left(2^2\right)}_{2} = \underbrace{2\cdot 1}_{2}$ ✓

(m=3) $\underbrace{\frac{1}{3}\left(2^3\right)}_{8/3} \neq 2\cdot 1$ ✗ $\longrightarrow \boxed{p=2}$

7 a) Using base functions, obtain an interpolatory num. diff formule to estimate $f'(z)$ using 4 distinct nodes: x_0, x_1, x_2, x_3 with $z \in [x_0, x_3]$ and such that it is of the highest possible order. Obtain its truncation error.

$z \in [x_0, x_3]$ but is not a node in this case



$x_0 = z-2h$

$x_1 = z-h$

$x_2 = z+h$

$x_3 = z+2h$

$$f(x) = \underbrace{\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}}_{L_0(x)} f(x_0) + \underbrace{\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}}_{L_1(x)} f(x_1) + \underbrace{\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}}_{L_2(x)} f(x_2) + \underbrace{\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}}_{L_3(x)} f(x_3)$$

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7 T. a)

- $L_0(x) = \frac{(x-z+h)(x-z-h)(x-z-2h)}{(-h)(-3h)(-4h)} \rightarrow L'_0(x) = \frac{-1}{12h^3}\left[(x-z-h)(x-z-2h)+(x-z+h)(x-z-2h)+(x-z-h)(x-z+h)\right]$

$\hookrightarrow L'_0(z) = \frac{-1}{12h^3}\left[\underbrace{(-h)(-2h)}_{2\cancel{h}^2} + \underbrace{(h)(-2h)}_{-2\cancel{h}^2} + \underbrace{(h)(-h)}_{-h^2}\right] = \boxed{\frac{1}{12h} = A_0}$

- $L_1(x) = \frac{(x-z+2h)(x-z-h)(x-z-2h)}{(h)(-2h)(-3h)} \rightarrow L'_1(x) = \frac{(x-z-h)(x-z-2h)+(x-z+2h)(x-z-h)+(x-z+2h)(x-z-2h)}{6h^3}$

$\hookrightarrow L'_1(z) = \frac{\overbrace{(-h)(-2h)}^{2\cancel{h^2}} + \overbrace{(2h)(-h)}^{-2\cancel{h^2}} + (2h)(-2h)}{6h^3} = \boxed{\frac{-2}{3h} = A_1}$

- $L_2(x) = \frac{(x-z+2h)(x-z+h)(x-z-2h)}{(3h)(2h)(-h)} \rightarrow L'_2(x) = \frac{(x-z+h)(x-z-2h)+(x-z+2h)(x-z-2h)+(x-z+2h)(x-z+h)}{-6h^3}$

$\hookrightarrow L'_2(z) = \frac{(h)(-2h)+(2h)(-2h)+(+2h)(h)}{-6h^3} = \boxed{\frac{2}{3h} = A_2}$

- $L_3(x) = \frac{(x-z+2h)(x-z+h)(x-z-h)}{(4h)(3h)(h)} \rightarrow L'_3(x) = \frac{(x-z+h)(x-z-h)+(x-z+2h)(x-z-h)+(x-z+2h)(x-z+h)}{12h^3}$

$\hookrightarrow L'_3(z) = \frac{(h)(-h)+(2h)\cancel{(-h)}+(2h)\cancel{(h)}}{12h^3} = \boxed{\frac{-1}{12h} = A_3}$

$f'(z) = A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)$

$\hookrightarrow f'(z) = \frac{1}{12h}\cdot f(z-2h) - \frac{2}{3h} f(z-h) + \frac{2}{3h} f(z+h) - \frac{1}{12h} f(z+2h) + E$

$\hookrightarrow \boxed{f'(z) = \frac{f(z-2h) - 8f(z-h) + 8f(z+h) - f(z+2h)}{12h} + E}$

$$E = e'(x) = f'[x_0, x_1, x_2, x_3, z]\,\pi(x) + f[x_0, x_1, x_2, x_3, z]\,\pi'(x)$$

$$\begin{cases} \pi(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3) = (x-z+2h)(x-z+h)(x-z-h)(x-z-2h) \\ \pi'(x) = (x-z+h)(x-z-h)(x-z-2h) + (x-z+2h)(x-z-h)(x-z-2h) + (x-z+2h)(x-z+h)(x-z-2h) + (x-z+2h)(x-z+h)(x-z-h) \end{cases}$$

$$\boxed{x=z} \rightarrow \begin{cases} \pi(z) = (2h)(h)(-h)(-2h) = 4h^4 \\ \pi'(z) = (h)(-h)\cancel{(-2h)} + (2h)\cancel{(-h)}(-2h) + (2h)\cancel{(h)}(-2h) + (2h)\cancel{(h)}(-h) = 0 \end{cases}$$

$$E = f'[x_0, x_1, x_2, x_3, z]\,\pi(z) = f[x_0, x_1, x_2, x_3, z, z]\cdot 1!\;\pi(z)$$

$$\hookrightarrow \boxed{E = \frac{f^{5)}(\xi)\cdot 4h^4}{5!} = \frac{f^{5)}(\xi)\,h^4}{30}}$$

b) $v(t) = 0'98\cdot i'(t) + 0'142\, i(t)$; approximate $v(1'02)$

h h h h

t	1	1'01	1'02	1'03	1'04
i(t)	3'1	3'12	3'14	3'18	3'24

$$\begin{cases} h = 0'01 \\ z = t = 1'02 \end{cases}$$

$$i'(1'02) = \frac{i(1) - 8i(1'01) + 8i(1'03) - i(1'04)}{12\cdot 0'01} = \frac{3'1 - 24'96 + 25'44 - 3'24}{0'12} = \frac{0'34}{0'12}$$

$$\boxed{V(1'02) = 0'98\,\frac{0'34}{0'12} + 0'142\cdot 3'14 = 3'2225}$$

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1 $P(x) = \boxed{4} + \boxed{3}(x+1) \boxed{-2}(x+1)^2 + \boxed{\frac{3}{2}}(x+1)^2(x-1) \boxed{-\frac{1}{2}}(x+1)^2(x-1)^2$

$x_0 = -1, \quad x_1 = 1$

a) what interp. data has been used to calculate the pol?

As we can see, we have both nodes more than once.

$x_0 = -1$ → we have it DOUBLE → we need $f(x_0)$ and $f'(x_0)$

$x_1 = 1$ → we have it TRIPLE → we need $f(x_1)$, $f'(x_1)$, $f''(x_1)$

b) table of differences.

x_i	$f_{i,0}$	$f_{i,1}$	$f_{i,2}$	$f_{i,3}$	$f_{i,4}$
-1	4	—	—	—	—
-1	4	3	—	—	—
1	2	-1	-2	—	—
1	2	1	1	1'5	—
1	2	1	2	0'5	$-0'5$

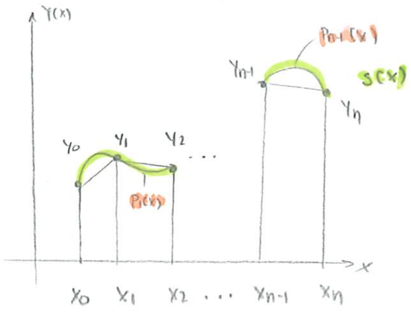
- $\boxed{f(x_0) = f_{0,0} = 4}$
- $\boxed{f'(x_0) = 3}$

- $f_{2,0} = f_{3,0} = f_{4,0} = f(x_1)$
- $f_{3,1} = f_{4,1} = f'(x_1)$
- $f_{4,2} = f''(x_i)/2$

- $-0'5 = \frac{f_{4,3} - 1'5}{2} \rightarrow f_{43} = 0'5$
- $1'5 = \frac{f_{32} + 2}{2} \rightarrow f_{32} = 1$
- $0'5 = \frac{0'5 \cdot f''(x_1) - 1}{2} \rightarrow \boxed{f''(x_1) = 4}$
- $-2 = \frac{f_{21} - 3}{2} \rightarrow f_{21} = -1$
- $1 = \frac{f'(x_1) + 1}{2} \rightarrow \boxed{f'(x_1) = 1}$
- $-1 = \frac{f(x_1) - 4}{2} \rightarrow \boxed{f(x_1) = 2}$

2. T a) DEFINE what a cubic spline of a function $f(x)$ for a set of nodes : $x_0, x_1, \dots, x_n$ is. What conditions are added in the case of natural cubic splines and cubic splines with boundary cond?

Cubic splines are the solution to the Runge effect.



* our set of interp. points will have shorted and distinct nodes such that: $x_0 < x_1 < x_2 \dots x_{n-1} < x_n$

A spline of order 3 is a piecewise pol. function $s(x)$ that satisfies:

- $s(x_i) = f(x_i)$ $(i = 0, 1, \dots, n)$ (PIECEWISE APPROX = ACTUAL POL.)
- $s(x) \Big|_{[x_i, x_{i+1}]} = P_i(x) \in \mathbb{P}_3$
- $s \in C^2(\mathbb{R})$ (SMOTHNESS)

* AS a consequence, from x_1 to x_{n-1} the following must be verified

$$\underbrace{\begin{cases} P_i(x_{i+1}) = P_{i+1}(x_{i+1}) \\ P_i'(x_{i+1}) = P_{i+1}'(x_{i+1}) \\ P_i''(x_{i+1}) = P_{i+1}''(x_{i+1}) \end{cases}}_{\text{smothness}} \quad (i = 0, 1, \dots, n-2)$$

- number of conditions :

$$\begin{array}{ll} 2n & (\text{INTERP}) \\ n-1 & (C^1) \\ n-1 & (C^2) \\ \hline \boxed{4n - \underbrace{2}_{\to 2\text{dof}}} & \end{array}$$

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2. a)

NATURAL CUBIC SPLINE

$$\begin{cases} S''(x_0) = P_0''(x_0) = 0 \\ S''(x_n) = P_{n-1}''(x_n) = 0 \end{cases}$$

CUBIC SPLINE WITH BOUNDARY CONDITIONS

$$\begin{cases} S'(x_0) = P_0'(x_0) = f_0'(x_0) \\ S'(x_n) = P_{n-1}'(x_n) = f'(x_n) \end{cases}$$

b) NAT. CUBIC SPLINE

$$S(x) = \begin{cases} 1 + a(x-1) \; . \; b(x-1)^3 & 1 \leq x < 2 \\ 1 + c(x-2) - (3/4)(x-2)^2 + d(x-2)^3 & 2 \leq x \leq 3 \end{cases}$$

* knowing that $S(x)$ interpolates the data $\underset{x_0}{(1,1)}$, $\underset{x_1}{(2,1)}$ and $\underset{x_n}{\underset{\shortparallel}{\underset{x_2}{(3,0)}}}$ → obtain a, b, c, d

NAT. CUBIC SPLINE CONDITIONS

$$S'(x) = \begin{cases} a - 3b(x-1)^2 \\ c - 3/2\,(x-2) + 3d(x-2)^2 \end{cases} \longrightarrow S''(x) = \begin{cases} -6b(x-1) \\ -1'5 + 6d(x-2) \end{cases}$$

$$\begin{cases} S''(1) = 0 \rightarrow -6b(1-1) = 0 \rightarrow \forall b \\ S''(3) = 0 \rightarrow -1'5 + 6d = 0 \rightarrow \boxed{d = 0'25} \end{cases}$$

DATA CONDITIONS

$S(1) = 1 \rightarrow 1 + 0 + 0 = 1$ ✓ $\forall a, b \in \mathbb{R}$

$S(2) = 1 \rightarrow 1 + (0) + 0 + 0 = 1$ ✓ $\forall c, d \in \mathbb{R}$

$S(3) = 0 \rightarrow 1 + c - 0'75 + \underset{0'25}{\underset{\shortparallel}{d}} = 0 \rightarrow \boxed{c = -0'5}$

SMOTHNESS

SEC¹ : $\overbrace{a - 3b(x_1 - 1)^2}^{P_0'(x_1)} = \overbrace{c - 1'5\cancel{(x_1-2)} + 3d\cancel{(x_1-2)}}^{P_1'(x_1)} \rightarrow a - 3b = c = -0'5 \rightarrow \boxed{a = 0'25}$

SEC² : $\underbrace{-6b(x_1 - 1)}_{P_0''(x_1)} = \underbrace{-1'5 + 6d\cancel{(x_1-2)}}_{P_1''(x_1)} \rightarrow \cancel{+}6b = \cancel{+}1'5 \rightarrow \boxed{b = 0'25}$

3. Knowing that Hermite interp. pol. verify:

$H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x)$ with $H_0(x) = 1$, $H_1(x) = 2x$

and that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \sqrt{\pi}$, find the Gauss Quadrature rule of order 5 or more with the fewest possible nodes

$N = 2n+1 = 5 \rightarrow n = 2 \rightarrow nn = 3$ (nodes) $\rightarrow$ 3 weights

$\boxed{H_3(x) = 2x\,H_2(x) - 2\,H_1(x)}$

- $H_2(x) = 2x\,H_1(x) - 2\cdot 1\,H_0(x) = 2x(2x) - 2\cdot 1 = 4x^2 - 2$

↳ $H_3(x) = 2x(4x^2-2) - 2\cdot(2x) \quad 8x^3 - 12x$

$H_3(x) = 0 \rightarrow 4x(2x^2-3) = 0 \;\begin{cases} x = 0 \\ 2x^2 = 3 \end{cases} \rightarrow \boxed{X_0 = -\sqrt{1'5},\; X_1 = 0,\; X_2 = \sqrt{1'5}}$

$f(x) \equiv 1 \rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} = A_0\cdot f(x_0) + A_1\cdot f(x_1) + A_2\,f(x_2) \rightarrow \boxed{A_1 + A_2 + A_3 = \sqrt{\pi}}$

$f(x) \equiv x \rightarrow \int_{-\infty}^{\infty} x e^{-x^2}\,dx = 0 = A_0(-\sqrt{1'5}) + A_2(\sqrt{1'5}) \rightarrow \boxed{A_0 = A_2}$

↳ impar

$f(x) \equiv x^2 \rightarrow \int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} = A_0\cdot 1'5 + A_2\,1'5 \rightarrow \boxed{A_0 + A_2 = \sqrt{\pi}/3}$

$\boxed{A_0 = A_2 = \sqrt{\pi}/6}$

↳ $\boxed{A_1 = \sqrt{\pi} - \frac{2\sqrt{\pi}}{6} = 4\sqrt{\pi}/6}$

$$\boxed{Q = \frac{\sqrt{\pi}}{6}\left(f(-\sqrt{1'5}) + 4f(0) + f(\sqrt{1'5})\right)}$$

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4 $I = \int_0^1 e^{e^x} dx$ with a precision of at least 0'05 by means of the compound simpson Rule.

$f^{iv)}(x) = e^{e^x}(e^x + 7e^{2x} + 6e^{3x} + e^{4x})$ for $f(x) = e^{e^x}$

- SIMPLE SIMPSON'S RULE : $\boxed{\frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))}$

↳ we don't know yet the amount of intervals we need.

- CLOSED NEWTON COTE → $\overbrace{nn = 3 \longrightarrow N = 3}^{nn\,(impar) = N}$; nodes $x_0 = -h$, $x_1 = 0$, $x_2 = h$

(impar)

↳ 1ST moromial without exact integration : $f(x) = x^4$

- $f(x) = x^4 \to f'(x) = 4x^3 \to f''(x) = 12x^2 \to f'''(x) = 24x \to f^{iv)}(x) = 24$

↳ $\int_{-h}^{h} x^4 dx = \left.\frac{x^5}{5}\right]_{-h}^{h} = \frac{2h^5}{5} = Q_{ss} + E_{ss}$

$$\frac{2h^5}{5} = \underbrace{\frac{h}{3}\cdot(2h^4)}_{2h^5/3} + E \longrightarrow E = K \cdot f^{iv)} = 24K$$

$\neq$

↳ $24K = \frac{2h^5}{5} - \frac{2h^5}{3} = \frac{(6-10)h^5}{15} \longrightarrow \boxed{K = \frac{-4h^5}{24 \cdot 15} = -\frac{h^5}{90}}$

*once we have define it for nn=3 → we are generalizing it for m

where $\boxed{M = \frac{(b-a)}{2h}}$

$$E_{SS} = \frac{-h^5}{90} \cdot f^{(IV)}(\xi) \longrightarrow E_c = -\frac{h^4(b-a)}{180} \cdot f^{(IV)}(\xi)$$

UPPERBOUND

$$\cdot\ |E_c| < 0'05 \rightarrow \left| \frac{-h^4(b-a)}{180} f^{(IV)}(\xi) \right| \leq \frac{h^4(b-a)}{180} \cdot M = \frac{h^4(1-0)}{180} \cdot 3478'707 = 19'326h^4$$

$\uparrow$ (el 2 está ahí)

esa $M \neq M$ subint

$$\cdot\ f^{(IV)}(1) = e^{e'}(e' + 7e^2 + 6e^3 + e^4) = 3478'707$$

$$\hookrightarrow 19'326h^4 \leq 0'05 \rightarrow h \leq 0'225531 \rightarrow 0'225531 = \frac{1}{M} \rightarrow = 4'434 \rightarrow \underline{M=5} \text{ subintervals}$$

$$h = \frac{1}{10} = 0'1$$

SUBINTERVALS !!

$$Q_c = \frac{h}{3}\left[\overbrace{f(x_0) + 4(x_1) + 2f(x_2)}^{M=1} + \overbrace{4f(x_3) + 2f(x_4)}^{M=2} + \overbrace{4f(x_5) + 2f(x_6)}^{M=3} + \overbrace{4f(x_7) + 2f(x_8)}^{M=4} + \overbrace{4f(x_9) + f(x_{10})}^{M=5} \right]$$

$$\hookrightarrow \boxed{Q_c} = \frac{0'1}{3}\left[e + 4\cdot\underbrace{(e^{e^{0'1}} + e^{e^{0'3}} + e^{e^{0'5}} + e^{e^{0'7}} + e^{e^{0'9}})}_{31'26849181} + 2\underbrace{(e^{e^{0'2}} + e^{e^{0'4}} + e^{e^{0'6}} + e^{e^{0'8}})}_{23'28058062} + e^e \right] = \boxed{6'316922418}$$

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5 For the Euler multistep methods:

Euler (Adams-Bashforth $p=0$, $m=0$)

Trapezoidal Rule (Adams-Moulton $p=0$, $m=1$) $\quad y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})$

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5. a) Obtain the absolute stability region and represent them graphically

EULER

$y_{n+1} = y_n + f_n \cdot h$

0) $k = (n+1) - n = 1$

1) $y_{n+k} = -\sum_{n=0}^{k-1} \alpha_j y_{n+j} + h \sum_{i=0}^{k} \beta_i f_{n+i} \rightarrow y_{n+1} = -\alpha_0 \cdot y_n + h(\beta_0 f_n + \beta_1 f_{n+1})$

$\hookrightarrow \alpha_0 = -1 \ ; \ \alpha_1 = 1$

$\hookrightarrow \beta_0 = 1 \ ; \ \beta_1 = 0$

2) $\rho(z) = \sum_{j=0}^{k} \alpha_j \cdot z^j$ // $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$

$\rho(z) = -1 \cdot z^0 + 1 \cdot z^1 \rightarrow \rho(z) = z - 1$

$\sigma(z) = 1 \cdot z^0 + 0 \cdot z^1 \longrightarrow \sigma(z) = 1$

3) $\rho(1) = 0$ // $\rho'(1) = \sigma(1)$

$\rho(1) = 1 - 1 = 0$ ✓

$\rho'(1) = 1 = \sigma(1) = 1$ ✓

4) ROOTS of $\rho(z)$

$\hookrightarrow z - 1 = 0 \rightarrow z = 1$ simple

5) STABILITY REGION:

$\boxed{\pi(z) = \rho(z) - \bar{h}\,\sigma(z)}$

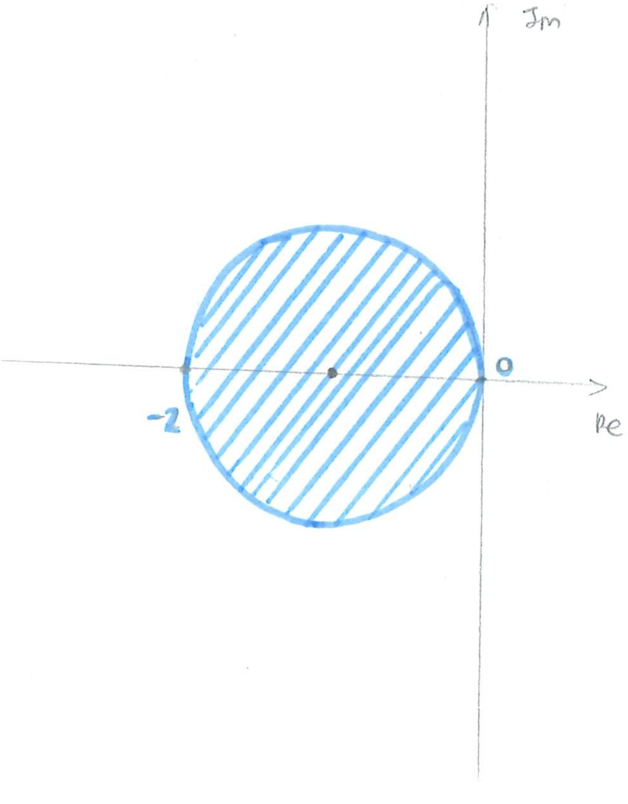
$\pi(z) = z - 1 - \bar{h} = 0$ $(z(\bar{h}))$

$\hookrightarrow \boxed{z = \bar{h} + 1}$

$|z(\bar{h})| < 1$

$\hookrightarrow 1 + \bar{h} < 1 \rightarrow \boxed{\bar{h} < 0}$

$\hookrightarrow -1 - \bar{h} < 1 \rightarrow \boxed{\bar{h} > -2}$



TRAPEZOIDAL RULE

$$\boxed{y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})}$$

0) $k = (n+1) - n = 1$

1) $y_{k+n} = -\sum_{i=0}^{k-1}\alpha_i y_{n+i} + h\sum_{i=0}^{k}\beta_i f_{n+i} \longrightarrow y_{n+1} = -\alpha_0 y_n + h(\beta_0 f_n + \beta_1 f_{n+1})$

$\hookrightarrow \alpha_0 = -1 \; ; \; \alpha_1 = 1$

$\hookrightarrow \beta_0 = 1/2 ; \; \beta_1 = 1/2$

2) $\rho(z) = \sum_{i=0}^{k}\alpha_i \cdot z^i \; // \; \sigma(z) = \sum_{i=0}^{k}\beta_i z^i$

$\rho(z) = -1 \cdot z^0 + 1 \cdot z \rightarrow \rho(z) = z - 1$

$\sigma(z) = \frac{1}{2}z^0 + \frac{1}{2}z \rightarrow \sigma(z) = 0'5 + 0'5z$

3) $\rho(1) = 0 \; // \; \rho'(1) = \sigma(1)$

$\rho(1) = 1 - 1 = 0$ ✓

$\rho'(1) = 1 = 0'5 + 0'5 = 1$ ✓

4) ROOTS $\rho(z) \rightarrow z - 1 = 0 \rightarrow z = 1$ SIMPLE ✓

5) STABILITY REGION

$\pi(z) = \rho(z) - \bar{h}\sigma(z)$

$\pi(z) = z - 1 - \frac{\bar{h}}{2}(z+1) = 0$

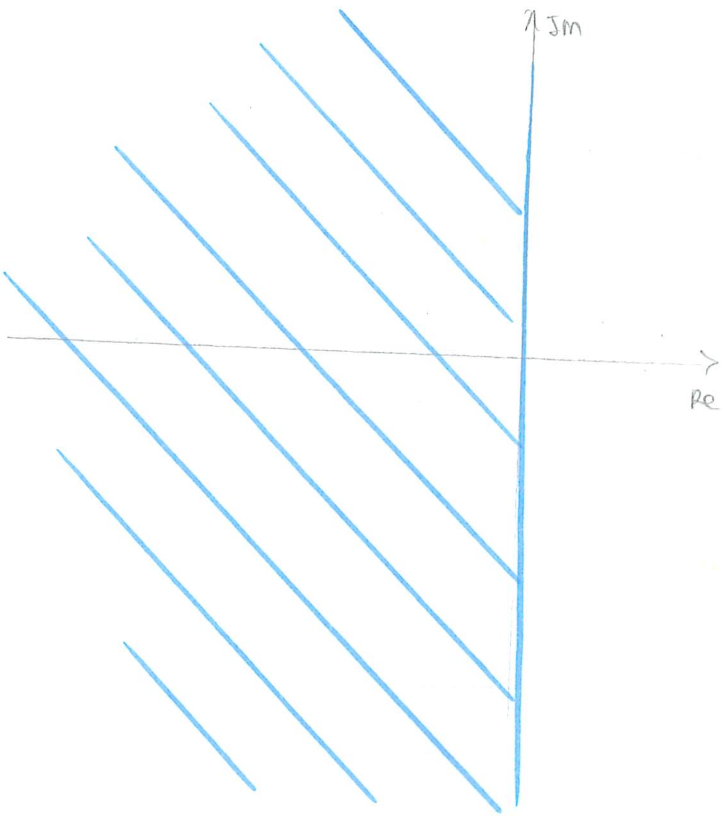
$\hookrightarrow z(1 - \bar{h}/2) = \frac{\bar{h}}{2} + 1$

$\hookrightarrow \boxed{z = \frac{\bar{h}/2 + 1}{1 - \bar{h}/2}}$

$|z(\bar{h})| < 1 \rightarrow |\bar{h} + 2| < |2 - \bar{h}| = |\bar{h} - 2|$

$|\bar{h} - (-2)| < |\bar{h} - 2|$

$\hookrightarrow \forall \bar{h} < 0$



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5 b) for the system: $\begin{cases} y' = -y + z \\ z' = e^{-x} - y/2 \end{cases}$ determine the values of h that guarantee the weak stability of Euler's method

- S.ODE'S → $\begin{pmatrix} y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 \\ -1/2 & 0 \end{pmatrix}}_{J} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-x} \end{pmatrix}$

- $J = \begin{pmatrix} -1 & 1 \\ -1/2 & 0 \end{pmatrix} \rightarrow |J - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 1 \\ -1/2 & -\lambda \end{vmatrix} = (1+\lambda)\lambda + \frac{1}{2} = 0 \rightarrow \lambda^2 + \lambda + 0'5 = 0$$

$$\hookrightarrow \lambda = \frac{-1 \pm \sqrt{1-2}}{2} = -\frac{1}{2} \pm \frac{i}{2} = \text{Eigs}(J)$$

- $\text{Eigs}(hJ) = h \cdot \text{Eighs}(J)$

$$\hookrightarrow h \cdot \left(\frac{-1}{2} \pm \frac{i}{2} \right) = \text{Eigs}(hJ)$$

la circunf tiene su centro en el -1

$$\underbrace{\left(\frac{-h}{2} - (-1) \right)^2 + \left(\frac{\pm h}{2} \right)^2 = 1^2}_{\text{STABILITY REGION}}$$

$$\hookrightarrow \frac{h^2}{4} - h + \cancel{1} + \frac{h^2}{4} = \cancel{1} \rightarrow h^2 - 2h = 0 \begin{cases} h = 0 \\ \boxed{h = 2} \end{cases}$$

c.) $\begin{cases} y' = -y + z \\ z' = e^{-x} - y/2 \end{cases}$ $\quad y(0) = 2$ $\quad z(0) = 1$

using Heun's predictor - corrector (Pred: Euler // Corr: Trap) in the interval [0, 0'2] using a P(EC)^{s}E scheme with step size h=0'1 and stopping crit. of 0'5% over the whole interval.

- $\underset{\sim}{Y} = \begin{pmatrix} y \\ z \end{pmatrix} \longrightarrow \underset{\sim}{Y}' = \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} -y+z \\ e^{-x} - y/2 \end{pmatrix} = f(y, z)$

$x_0 = 0, \quad x_1 = 0'1, \quad x_2 = 0'2$

$\longrightarrow \boxed{K=0} \quad \underset{\sim}{Y}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

* PREDICTION: EULER $\left(Y_1 = Y_0 + \overset{x=0}{f_0} \cdot h \right)$

$$\underset{\sim}{Y}_{1_P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2+1 \\ 1-1 \end{pmatrix} \cdot 0'1 = \begin{pmatrix} 1'9 \\ 1 \end{pmatrix}$$

* CORRECTION: TRAP. RULE $\left(Y_1 = Y_0 + \frac{h}{2} \left(\overset{x=0}{f_0} + \overset{x=1}{f_1} \right) \right)$

$$\underset{\sim}{Y}_{1_c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 0'05 \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1'9+1 \\ e^{-0'1} - 1'9/2 \end{pmatrix} \right] = \begin{pmatrix} 1'905 \\ 0'99774187 \end{pmatrix}$$

$$\% \text{ CRIT} = \max \left(\frac{|Y_{corr} - Y_{pred}|}{Y_{corr}} \right) \cdot 100$$

$$\hookrightarrow \% = \left(\frac{0'005}{1'905} \right) \cdot 100 = 0'262467191\,\% < 0'5\,\%$$

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5

$\boxed{k=1}$ $Y_1 = \begin{pmatrix} 1'905 \\ 0'9977412 \end{pmatrix}$

* PREDICTION : EULER $(Y_2 = Y_1 + f_1 h)$

$$\underset{\sim}{Y_{2p}} = \begin{pmatrix} 1'905 \\ 0'997742 \end{pmatrix} + 0'1 \begin{pmatrix} -0'907258 \\ -0'047662581 \end{pmatrix} = \begin{pmatrix} 1'8142742 \\ 0'9929757 \end{pmatrix}$$

* CORRECTION: TRAP. RULE $\left(Y_2 = Y_1 + \frac{h}{2} + (\overset{x=0'1}{f_1} + \overset{x=0'2}{f_2})\right)$

$$\underset{\sim}{Y_{2c}} = \begin{pmatrix} 1'905 \\ 0'997742 \end{pmatrix} + 0'05 \left[\begin{pmatrix} -0'907258 \\ -0'047662581 \end{pmatrix} + \begin{pmatrix} -0'8212985 \\ -0'088406346 \end{pmatrix} \right] = \begin{pmatrix} 1'81857217 5 \\ 0'99093855 \end{pmatrix}$$

$$\% = \frac{0'004297975}{1'81857217 5} \cdot 100 = 0'2363\ \% < 0'5\%\ ✓$$

SOLUTION : $\boxed{\underset{\sim}{Y_0} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} ;\ \underset{\sim}{Y_1} = \begin{pmatrix} 1'905 \\ 0'997742 \end{pmatrix} ;\ \underset{\sim}{Y_2} = \begin{pmatrix} 1'818572 \\ 0'990938 \end{pmatrix}}$

6. $$f'(t) = \underbrace{\frac{-25f(t) + 48f(t+h) - 36f(t+2h) + 16f(t+3h) - 3f(t+4h)}{12h}}_{D} + \underbrace{\frac{h^4}{5} f^{5)}(\xi)}_{E}$$

$f(x) = e^x + \sin(x)$ $x \in [0,2]$ evaluated with inaccuracies of up to 10^{-8}

Calculate the optimal step size $h = h_{opt}$ for the num. diff. formula

UPPER BOUND

$M_1 = f(2) = 8'2983$

UPPER BOUND (M)

1. $$|E| = \left| \frac{h^4 f^{5)}(\xi)}{5} \right| \leq \frac{h^4 \cdot 8'29835}{5} \doteq g_1(h)$$

2. $$|AF| = \sum_{i=0} |A_i| = \frac{1}{12h} \cdot (25+48+36+16+3) = \frac{32}{3h}$$

3. $$|E_r| = \varepsilon |AF| = \frac{10^{-8} \cdot 32}{3h} \doteq g_2(h)$$

4. $$g(h) = \frac{h^4 \cdot 8'29835}{5} + \frac{32 \cdot 10^{-8}}{3h} = 1'65967 h^4 + \frac{32 \cdot 10^{-8}}{3} h^{-1}$$

5. $$g'(h) = 6'63868 h^3 - \frac{32 \cdot 10^{-8}}{3} \frac{1}{h^2} = 0$$

$$\hookrightarrow 0'622376 h^3 = \frac{10^{-8}}{h^2} \longrightarrow h^5 = 16'06744 \cdot 10^{-9} \longrightarrow \boxed{h_{opt} = 0'027617817}$$

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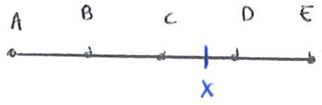
ORD.

1. $L = 10m$

5 equally spaced points

$X = 7$

0	2'5	5	7'5	10
A	B	C	D	E
1'61534	2'18174	2'25203	1'98124	1'56423



X is at 2m from the pipe's midpoint

From the estimation of truncation errors, decide what is better: quadratic or cubic interpolation.

	$\Delta_{0,i}$	$\Delta_{1,i}$	Δ_{i2}	Δ_{i3}	Δ_{i4}
0	1'61534	—	—	—	—
2'5	2'18174	0'5664	—	—	—
5	2'25203	0'07029	-0'49611	—	—
X → 7'5	1'98124	-0'27079	-0'34108	0'15503	—
10	1'56423	-0'41701	-0'14622	0'19486	0'03983

SER
X

$$\begin{cases} P_2(X) = 2'25203 - 0'27079(X-5) - 0'14622(X-5)(X-7'5) \\ P_3(X) = 2'18174 + 0'07029(X-2'5) - 0'34108(X-2'5)(X-5) + 0'19486(X-2'5)(X-5)(X-7'5) \end{cases}$$

$$\boxed{e_2(X) = \underbrace{f[C,D,E,X]}_{\Delta^{3)}(3)/3!\,h^3}\Pi(X)} \longrightarrow e_2(X) = \frac{\overset{\text{0'19486 (NEXT TERM)}}{\Delta^{3)}(B)}}{3!\cdot h^3}(X-5)(X-7'5)(X-10)$$

$$e_3(X) = \frac{\overset{0'03983}{\Delta^{4}(A)}}{4!\,h^4}(X-2'5)(X-5)(X-7'5)(X-10)$$

$$\boxed{e_2(7)} = \frac{0'19486}{\cancel{6}\cdot(2'5)^3}\,\cancel{2}\cdot(\cancel{-}0'5)(\cancel{-}3) = \boxed{0'00623552}$$

$$\boxed{e_3(7)} = \frac{0'03983}{\cancel{24}_4\;\cancel{2}'5^4}\cdot 4'5\cdot\cancel{2}\cdot 0'5\cdot\cancel{3} = \boxed{0'000573552}$$

$\longrightarrow e_3(7) < e_2(7)$

↳ CUBIC INTERPOLATION WILL BE BETTER

b) use the previous interp chosen and evaluate it optimally.

* AS WE ARE USING B, C, D, E WE HAVE TO CARRY OUT A CHANGE OF VARIABLE:

$$\boxed{x = x_0 + th} \longrightarrow x = 2'5 + 2'5t \longrightarrow \left(t = \frac{7 - 2'5}{2'5} = 1'8\right)$$

$$q_3(t) = 2'18174 + \frac{0'07029t}{1!} - \frac{0'34108\,t(t-1)}{2!} - \frac{0'19486}{3!}\,t(t-1)(t-2)$$

$$\hookrightarrow q_3(t) = 2'18174 + t\left[0'07029 + \frac{(t-1)}{2}\left(-0'34108 + \frac{0'19486}{3}(t-2)\right)\right] \qquad \left(t = 1'8\right)$$

$-0'012990666$

$-0'354070666$

$-0'141628266$

$-0'071338266$

$-0'12840888$

$2'05333112$

$$\boxed{q_3(1'8) = 2'05333112}$$

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2. a) We know cubic spline with boundary conditions determined by $x_0 < x_1 < x_2 < \dots < x_n$ and $f(x_0), f(x_1), \dots, f(x_n), f'(x_0), f'(x_n)$ is optimal. Explain in what sense and state precisely the corresponding theoretical result.

A spline of order n is a piecewise pol. function that satisfies:

1. $S(x_i) = f(x_i)$ → PIECEWISE APPROX = ACTUAL APPROX
2. $S(x)\Big|_{[x_i, x_{i+1}]} = P_i(x) \in \mathbb{P}_n$
3. $S \in C^{n-1}(\mathbb{R})$ → SMOTHNESS

↳ coming from the last condition, we have that if our polynomial is of degree 3 (cubic):

$S \in C^2(\mathbb{R})$

↳ our number of conditions will be :

$$\begin{array}{ll} 2n & (\text{INTERP}) \\ n-1 & C^1 \\ n-1 & C^2 \\ \hline 4n - \underbrace{2}_{\text{DOF}} & \end{array}$$

so, as we see, we have 2 degrees of freedom that will be satisfied by the given conditions, such that:

$$\left\{\begin{array}{l} S'(x_0) = P_0'(x_0) = f'(x_0) \\ S'(x_n) = P_{n-1}'(x_n) = f'(x_n) \end{array}\right\} \text{DATA}$$

+

b) One wants to build a quadratic spline of class C^1 with nodes:

$x_0 < x_1 < \dots < x_n$ and the corresponding ordinates $y_0, y_1, y_2, \dots, y_n$.

Can it be done ensuring $f'(x_0) = f'(x_n) = 0$

· The first polynomial piece: $P_0(x)$ (degree ≤ 2) is determined by
$P_0(x_0) = y_0$
$P_0'(x_0) = 0$
$P_0(x_1) = y_1$

· Then for the 2nd one we will have
$P_1(x_1) = y_1$
$P_1'(x_1) = y_1'$
$P_1(x_2) = y_2$

⋮

And so on, until we reach $P_n(x)$

↳ It would be really difficult for $f'(x_0), f'(x_n)$ to be equal to zero.

3 $\int_0^{\pi} e^{\cos x}\,dx$; 0'5 % precision. use Gauss quadrature

*CHANGE OF VARIABLE

$$\begin{cases} \cos(x) = t \\ -\sin(x)\,dx = dt \longrightarrow dx = \dfrac{-dt}{\sin(x)} = \dfrac{-dt}{\sqrt{1-\cos^2(x)}} = \dfrac{-dt}{\sqrt{1-t^2}} \end{cases} \longrightarrow \underline{\text{GAUSS-CHEBYSHEV}}$$

* INTEGRATION LIMITS:

$\cos(\pi) = -1 = t_2$

$\cos(0) = 1 = t_1$

$$\longrightarrow \int_0^{\pi} e^{\cos(x)}\,dx = \int_1^{-1} \frac{-e^t \cdot dt}{\sqrt{1-t^2}} = \int_{-1}^{1} \frac{e^t\,dt}{\sqrt{1-t^2}}$$

$$\boxed{f(t) = e^t}$$

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3.

GAUSS - CHEBYSHEV
$$\begin{cases} t_i = \cos\left(\dfrac{\pi/2 + \pi i}{n+1}\right) \\ W = \dfrac{\pi}{n+1} \end{cases}$$

$\boxed{n=0}$
$$\begin{cases} W = \pi \\ t_0 = \cos(\pi/2) = 0 \end{cases} \longrightarrow \boxed{Q_1 = \pi \cdot e^0 = \pi}$$

$\boxed{n=1}$
$$\begin{cases} W = \pi/2 \\ t_{1,2} = \pm\sqrt{2}/2 \end{cases} \longrightarrow \boxed{Q_2 = \frac{\pi}{2}\left(e^{\sqrt{2}/2} + e^{-\sqrt{2}/2}\right) = 3'960266}$$

$$\hookrightarrow \%\varepsilon = \frac{|3'960266 - \pi|}{3'960266} \cdot 100 = \underline{20'6721\% > 0'5\%}$$

$\boxed{n=2}$
$$\begin{cases} W = \pi/3 \\ t_1 = 0'8660254 \\ t_2 = 0 \\ t_3 = -0'8660254 \end{cases} \longrightarrow \boxed{Q_3 = \frac{\pi}{3}\left(1 + e^{\sqrt{3}/2} + e^{-\sqrt{3}/2}\right) = 3'97732196}$$

$$\hookrightarrow \%\varepsilon = \frac{|3'97732196 - 3'960266|}{3'97732196} \cdot 100 = \%\ 0'42883 < 0'5\% \quad ✓$$

4 $\int_1^3 \left(\int_2^4 (7y^3 + y^2 x)\,dy \right) dx$ calculate it exactly using Newton-cotes (least possible comp cost)

Justify your choice

First I am changing the order of integration → f(x) is of a smaller degree than f(y)

$$\int_2^4 \left(\int_1^3 (7y^3 + y^2 x)\,dx \right) dy$$

$$A(x) = \int_1^3 (7y^3 + y^2 x)\,dx \rightarrow \boxed{N=1} \begin{cases} \text{ONC MIDPOINT RULE} \rightarrow \text{LESS COMPUTATIONAL COST} \\ \text{CNC TRAPEZOIDAL RULE} \end{cases}$$

$$\begin{cases} x_0 = \frac{3+1}{2} = 2 \\ h = 1 \end{cases}$$

$$A(x) = \int_1^3 (7y^3 + y^2 x)\,dx = 2h \cdot f(x_0) = 2(7y^3 + 2y^2) = 14y^3 + 4y^2$$

$$\int_2^4 (14y^3 + 4y^2)\,dy \longrightarrow \boxed{N=3} \begin{cases} \text{CNC} & \text{SIMPSON} \longrightarrow \text{least comp. cost} \\ \text{CNC} & \text{2ND SIMPSON} \\ \text{ONC} & \text{3 NODES} \end{cases}$$

$$\begin{cases} x_0 = 2 \\ x_1 = (4+2)/2 = 3 \\ x_3 = 4 \\ h = 1 \end{cases}$$

$$I = \int_2^4 (14y^3 + 4y^2)\,dy = \frac{h}{3} \cdot \left(f(x_0) + 4f(x_1) + f(x_2) \right)$$

$$I = \frac{1}{3}\Big(\underbrace{14 \cdot 8}_{112} + \underbrace{4 \cdot 4}_{16} + 4(\underbrace{14 \cdot 27}_{378} + \underbrace{4 \cdot 9}_{36}) + \underbrace{14 \cdot 64}_{896} + \underbrace{4 \cdot 16}_{64} \Big) = \boxed{\frac{2744}{3} = 914'667}$$

5

$$\begin{cases} x'(t) = x(t) - y(t) + t\,z(t) \\ y'(t) = -x(t) - y(t) + z(t) \\ z'(t) = t\,x(t) + y(t) - z(t) \end{cases} \qquad t=0 \rightarrow (1, 0, -1)$$

Use Enhahced Euler (Heun) method to timate its position at $t=0'1$ and $t=0'2$

($h=0'1$ stepsize)

$$r(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \longrightarrow r'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} x(t) - y(t) + t z(t) \\ -x(t) - y(t) + z(t) \\ t x(t) + y(t) - z(t) \end{pmatrix} = f(t)$$

$$r(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

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5 **ENHANCED EULER :** $$Y_{k+1} = Y_k + \frac{f(t_k, Y_k) + f(t_k+h_k, Y_k + f(t_k,Y_k)\cdot h_k)}{2} h_k$$

└ RK2

$$\begin{cases} k_1 = f(t_k, Y_k)\cdot h_k \\ k_2 = f(t_k+h_k, Y_k + k_1)\cdot h_k \end{cases} \longrightarrow Y_{k+1} = Y_k + \frac{k_1+k_2}{2}$$

(K=0) $t=0$, $r(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$K_1 = \begin{pmatrix} 1-0+0 \\ -1-0-1 \\ 0+0+1 \end{pmatrix} \cdot 0'1 = \begin{pmatrix} 0'1 \\ -0'2 \\ 0'1 \end{pmatrix}$$

$$K_2 = f\left(0'1, \begin{pmatrix} 1'1 \\ -0'2 \\ -0'9 \end{pmatrix}\right)\cdot 0'1 = \begin{pmatrix} 1'1 + 0'2 - 0'09 \\ -1'1 + 0'2 - 0'9 \\ 0'11 - 0'2 + 0'9 \end{pmatrix} 0'1 = \begin{pmatrix} 0'121 \\ -0'118 \\ 0'081 \end{pmatrix}$$

$$\boxed{r(0'1) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0'1 + 0'121 \\ -0'2 - 0'118 \\ 0'1 + 0'081 \end{pmatrix} \cdot 0'5 = \boxed{\begin{pmatrix} 1'1105 \\ -0'19 \\ -0'9095 \end{pmatrix}}}$$

(K=1) $t=0'1$, $r(0'1) = \begin{pmatrix} 1'1105 \\ -0'19 \\ -0'9095 \end{pmatrix}$

$$K_1 = \begin{pmatrix} 1'20955 \\ -1'83 \\ 0'83055 \end{pmatrix} 0'1 = \begin{pmatrix} 0'120955 \\ -0'183 \\ 0'083055 \end{pmatrix}$$

$$k_2 = f\left(0'2, \begin{pmatrix} 1'231455 \\ -0'373 \\ -0'826445 \end{pmatrix}\right)\cdot 0'1 = \begin{pmatrix} 0'1439166 \\ -0'16849 \\ 0'0699736 \end{pmatrix}$$

$$\boxed{r(0'2) = \begin{pmatrix} 1'1105 + 0'1324358 \\ -0'19 - 0'175745 \\ -0'9095 + 0'0765143 \end{pmatrix} = \boxed{\begin{pmatrix} 1'2429358 \\ -0'365745 \\ -0'8329857 \end{pmatrix}}}$$

6 Find a, b for: $y_n = Y_{n-2} + h[a f_n + b f_{n-3}]$ to be convergent with the maximum possible order. Explicit or implicit? How many steps?

$Y_{n+1} = Y_{n-1} + h[a f_{n+1} + b f_{n-2}]$

0) $\underline{\underline{K}} = (n)-(n-3) = \underline{\underline{3}}$ (NUMBER OF STEPS)

1) $Y_{n+3} = -\sum_{i=0}^{2} \alpha_i Y_{n+i} + h \sum_{i=0}^{3} \beta_i f_{n+i}$

$Y_{n+3} = -\alpha_0 Y_n - \alpha_1 Y_{n+1} - \alpha_2 Y_{n+2} + h[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3}]$

2)
$$\begin{cases} Y_n = Y_{n-2} + h[a f_n + b f_{n-3}] \\ Y_n = -\alpha_0 Y_{n-3} - \alpha_1 Y_{n-2} - \alpha_2 Y_n + h[\beta_0 f_{n-3} + \beta_1 f_{n-2} + \beta_2 f_{n-1} + \beta_3 f_n] \end{cases}$$

$$\begin{cases} \alpha_0 = 0 \\ \alpha_1 = -1 \\ \alpha_2 = 0 \\ \alpha_3 = 1 \end{cases} \qquad \begin{cases} \beta_0 = b \\ \beta_1 = \beta_2 = 0 \\ \beta_3 = a \end{cases}$$

3) $\rho(z) = \sum_{i=0}^{3} \alpha_i \cdot z^i$; $\sigma(z) = \sum_{i=0}^{3} \beta_i z^i$

- $\rho(z) = -z + z^3 \longrightarrow \rho'(z) = -1 + 3z^2$
- $\sigma(z) = b + az^3$

4) $\rho(1) = 0$; $\rho'(1) = \sigma(1)$

$\rho(1) = -1 + 1 = 0$ ✓

$\rho'(1) = -1 + 3 = b + a \longrightarrow \boxed{a + b = 2}$

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6 5) $\rho(z) = -z + z^3 \rightarrow |\text{roots } \rho| \leq 1$ (if root = 1 → simple)

$z(z^2-1)=0$ ↗ $z=0$ ↘ $z=1$, $z=-1$ ✓

6) $\frac{1}{m}\sum_{j=0}^{3} \alpha_j j^m = \sum_{j=0}^{3} \beta_j j^{m-1}$

$m=1$ $\quad \underbrace{\frac{1}{1}\left(-1\cdot 1^1 + 1\cdot 3^1\right)}_{2} = \underbrace{\left(b\cdot 0^0 + a\,3^0\right)}_{a+b}$ $\quad (2 = a+b)$ ✓

$m=2$ $\quad \underbrace{\frac{1}{2}(-1+9)}_{4} = 3a \longrightarrow \boxed{a = 4/3} \longrightarrow \boxed{b = 2/3}$

$m=3$ $\quad \frac{1}{3}(-1+27) = 9\cdot\frac{4}{3}$ $\quad (26/3 \neq 36/3) \longrightarrow \boxed{p=2}$ max. possible order

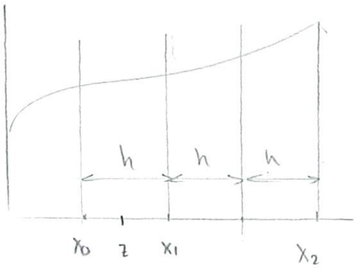
$$\boxed{y_n = y_{n-2} + \frac{2h}{3}\left[2 f_n + b f_{n-3}\right]}$$

↳ The METHOD is IMPLICIT

7. T a) Obtain a num. diff formula, as well as the error term

to estimate : $f'(z)$; x_0 , $x_1 = x_0 + h$, $x_2 = x_1 + 2h$ || $z = x_0 + 0'5h$

TAYLOR SERIES



$$\begin{cases} x_0 = x_0 \\ x_1 = x_0 + h \\ x_2 = x_0 + 3h \end{cases}$$

$$z = x_0 + 0'5h \longrightarrow x_0 = z - 0'5h \begin{cases} x_1 = z + 0'5h \\ x_2 = z + 2'5h \end{cases}$$

TAYLOR WAY (AD-HOC) : $f'(z) = D + E = \sum_{i=0}^{n} A_i f(x_i) + E$ $\underline{n=3}$

$$f'(z) = \sum_{i=0}^{3} A_i f(x_i) + E = A_0 \cdot f(x_0) + A_1 f(x_1) + A_2 f(x_2)$$

$$\hookrightarrow \boxed{f'(z) = A_0 \cdot f(z - 0'5h) + A_1 f(z + 0'5h) + A_2 f(z + 2'5h) + E}$$

$$f(z - 0'5h) = \frac{f(z)}{0!} + \frac{f'(z)}{1!} \cdot \left(-\frac{h}{2}\right) + \frac{f''(z)}{2!}\left(-\frac{h}{2}\right)^2 + \frac{f^{3)}(\xi)(-h/2)^3}{3!}$$

$$f(z + 0'5h) = f(z) + \frac{f'(z)(h/2)}{1!} + \frac{f''(z)(+h/2)^2}{2!} + \frac{f^{3)}(\xi)(+h/2)^3}{3!}$$

$$f(z + 2'5h) = f(z) + \frac{f'(z)(5h/2)}{1!} + \frac{f''(z)(5h/2)^2}{2!} + \frac{f^{3)}(\xi)(5h/2)^3}{3!}$$

$$f'(z) = f(z)(A_0 + A_1 + A_2) + f'(z)\left(-\frac{A_0 h}{2} + \frac{A_1 h}{2} + \frac{5h A_2}{2}\right) + \frac{f''(z)}{2}\left(\frac{A_0 h^2}{4} + \frac{A_1 h^2}{4} + \frac{A_2 25h^2}{4}\right) + \frac{f^{3)}(\xi)}{6}\left(-\frac{A_0 h^3}{8} + \frac{A_1 h^3}{8} + \frac{A_2 125h^3}{8}\right) + E$$

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7.

- $$f'(z) = \underbrace{(A_0+A_1+A_2)}_{0} f(z) + \underbrace{(-A_0+A_1+5A_2)}_{1} \frac{h f'(z)}{2} + \underbrace{(A_0+A_1+25A_2)}_{0} \frac{h^2 f''(z)}{8} + \underbrace{(-A_0+A_1+125A_2) \frac{h^3 f^{3)}(\xi)}{48}}_{0} + E$$

$$A_0 + A_1 + A_2 = 0 \rightarrow A_1 + A_0 = -A_2 \rightarrow A_0 = -A_1$$

$$-A_0 + A_1 + 5A_2 = \frac{2}{h} \rightarrow 2A_1 = \frac{2}{h} \rightarrow \boxed{A_1 = 1/h} \rightarrow \boxed{A_0 = -1/h}$$

$$A_0 + A_1 + 25A_2 = 0 \rightarrow \boxed{A_2 = 0}$$

$$E = -\frac{(2/h) \cdot h^3}{48} f^{3)}(\xi) = \boxed{-\frac{h^2 f^{3)}(\xi)}{24}}$$

$$\boxed{f'(z) = \underbrace{\frac{f(z+0'5h) - f(z-0'5h)}{h}}_{D} - \underbrace{\frac{h^2 f^{3}(\xi)}{24}}_{E}}$$

b) The formula obtained would give the exact value of

$f'(x_0 + 0'5h)$ for:

$$f(x) = \begin{cases} q(x) & x \le x_1 \quad (\text{degree } 2) \\ r(x) & x > x_1 \quad (\text{degree } 3) \end{cases}$$

- our aim is to obtain $f'(x_0 + 0'5h) \rightarrow x_0 + 0'5h = z < x_1 \rightarrow$ we are only working with $q(x)$ (DEGREE 2) $\rightarrow q^{3)}(x) = 0 \rightarrow E = 0$ ✓

YES

c) $f(x) = \ln(3x)$) $x_0 = 2$; $\varepsilon = 10^{-4}$ hopt?

$M \equiv$ UPER BOUND $= f^{3)}(2)$; $f' = \frac{1}{x} \rightarrow f'' = -\frac{1}{x^2} \rightarrow f''' = \frac{2}{x3} \rightarrow f^{3)}(2) = \frac{2}{8} = \underline{\underline{0'25 = M}}$

- $|E| = \left| -\frac{h^2 f^{3)}(\xi)}{24} \right| \leq \frac{h^2 0'25}{24} = g_1(h)$

- $|AF| = \sum_{i=0} |A_i| = \left|\frac{-1}{h}\right| + \left|\frac{1}{h}\right| + 0 = \frac{2}{h}$

- $|Er| = \varepsilon\, |AF| = \frac{2 \cdot 10^{-4}}{h} = g_2(h)$

- $g(h) = g_1(h) + g_2(h) \longrightarrow g(h) = \frac{0'25}{24} h^2 + 2 \cdot 10^{-4} h^{-1}.$

- $g'(h) = 0 \longrightarrow \frac{\cancel{2}\; 0'25}{24} h - \frac{\cancel{2} 10^{-4}}{h^2} = 0 \longrightarrow \frac{0'25 h^3}{24} = 10^{-4} \rightarrow h^3 = 0'0096$

$\hookrightarrow$ $\boxed{\text{hopt} = 212531713}$

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1

a) obtain the Taylor polynomial P_3 of degree ≤ 3 of $f(x) = \cos(2x)$ at $x_0 = 0$ from a table of differences, specifying the interp. data used, 6 sig. digits.

POLYNOMIAL ≤ 3 ⟶ we need x_0, x_1, x_2, x_3 but we only have $x_0 = 0$ so we need 3 extra conditions

$$f'(x) = -2\sin(2x) \longrightarrow f'(0) = 0$$

$$f''(x) = -4\cos(2x) \longrightarrow f''(0) = -4$$

$$f'''(x) = 8\sin(2x) \longrightarrow f'''(0) = 0$$

$$f(0) = 1$$

z_i	$f_{i,0}$	$f_{i,1}$	$f_{i,2}$	$f_{i,3}$
0	1	-	-	-
0	1	0	-	-
0	1	0	-2	-
0	1	0	-2	0

$$\boxed{P_3(x)} = 1 + 0 \cdot x - 2x^2 + 0 \cdot x^3 = \boxed{1 - 2x^2}$$

b) Write the general error term of osculating polynomials and particularize it for the polynomial obtained

$$\boxed{E = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot \Pi(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{k_0}(x-x_1)^{k_1} \cdot \ldots \cdot (x-x_n)^{k_n} \quad \text{f.s. } \xi \in C^n}$$

$$\boxed{E} = \frac{f^{(4)}(\xi)}{4!} \cdot x^4 = \frac{16\cos(2\xi)\,x^4}{4 \cdot 3 \cdot 2 \cdot 1} = \boxed{\frac{2\cos(2\xi)}{3} x^4}$$

c) Now we look for a pol. of degree ≤ 5 that also satisfies: $P_5(-1) = P_5(1) = -1$

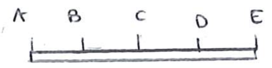
Clomplete the table above, find P_5 and its trunc. error.

z_i	$f_{i,0}$	$f_{i,1}$	$f_{i,2}$	f_{i3}	f_{i4}	$f_{i,5}$
0	1	—	—	—	—	—
0	1	0	—	—	—	—
0	1	0	-2	—	—	—
0	1	0	-2	0	—	—
-1	-1	2	-2	0	0	—
1	-1	0	-2	0	0	0

$$\boxed{P_5(x) = 1 + 0 \cdot x - 2x^2 + 0x^3 + 0x^4 + 0x^5 = 1 - 2x^2 \equiv P_3(x)}$$

$$\boxed{E = \frac{f^{6)}(\xi)}{6!} x^4 (x+1)(x-1) = \frac{f^{6)}(\xi)\, x^4 (x^2-1)}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{f^{6)}(\xi)\, x^4 (x^2-1)}{720}}$$

2



5 equally - spaced points

	2'5	5	7'5	
A (x=0)	B	C	D	E (x=10)
1'61534	2'18174	2'25203	1'98124	1'56423

There seens to be an intermediate point with max. signal. Estimate it by optimally evaluate

x_i	1'25	3'75	6'25	8'75
$f'(x_i)$	0'5664	0'07029	-0'27079	-0'41701

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2 maximal signal ⟶ $f'(x_i) = 0 = y$

y_i	$x_{i,0}$	$x_{i,1}$	$x_{i,2}$	$x_{i,3}$
0'5664	1'25	—	—	—
0'07029	3'75	-5'0392	—	—
-0'27079	6'25	-7'32966	2'73589	—
-0'41701	8'75	-17'0975	20'0448	-17'6009

$$x_i = 1'25 - 5'0392\,(y-0'5664) + 2'73589\,(y-0'07029)(y-0'5664) - 17'6009\,(y-0'5664)\,(y-0'07029)(y+0'27079)$$

In order to evaluate it optimally:

$$x_i = \underbrace{1'25 + \underbrace{(y-0'5664)\Big[\underbrace{-5'0392 + \underbrace{(y-0'07029)\big(\underbrace{2'73589 - \underbrace{17'6009\,(y+0'27079)}_{4'76615}}_{-2'03026}\big)}_{0'142707}}_{-4'89649}\Big]}_{2'77337}}_{4'02337} \qquad y=0$$

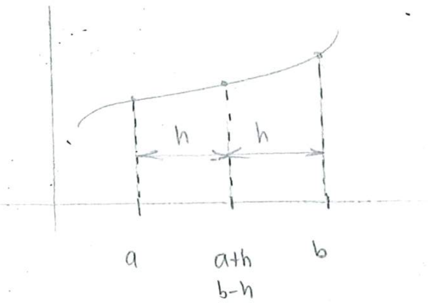
$$\boxed{x = 4'02337}$$

T

3. Derive simpson's simple quadrature rule by integrating an interp. pol. and obtain its truncation error term.

our goal: $Q_{simp} = \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)$

↳ we know we need 3 nodes



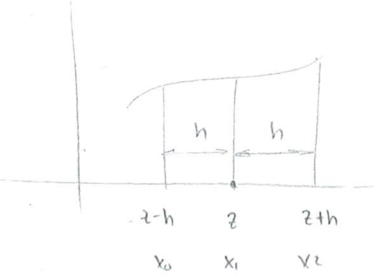
Interp. polynomial por 3 nodes :

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$Q = \int_{x_0}^{x_2} P_2(x)\,dx = \underbrace{\int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot f_0\,dx}_{W_0(x)} + \underbrace{\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1\,dx}_{W_1(x)} + \underbrace{\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2\,dx}_{W_2(x)} =$$

$$= W_0(x)\cdot f_0 + W_1(x)\cdot f_1 + W_2(x)\cdot f_2$$

* La he uado → más facil si : $x_1=0,\ x_2=h,\ x_0=-h$



$$\bullet\ \boxed{W_0} = \int_a^b \frac{(x-z)(x-b)}{(z-h-z)(z-h-z-h)}\,dx = \frac{1}{2h^2}\int_a^b (x-z)(x-b)\,dx = \frac{1}{2h^2}\int_a^b (x^2-bx-zx+bz)\,dx =$$

$$= \frac{1}{2h^2}\left[\frac{x^3}{3} - (b+z)\frac{x^2}{2} + bzx\right]_a^b = \frac{1}{2h^2}\left[\frac{b^3-a^3}{3} - (b+z)\left(\frac{b^2-a^2}{2}\right) + bz(b-a)\right] =$$

$$= \frac{1}{2h^2}\left[\frac{6hz^2+2h^3}{3} - \frac{\cancel{4}^{2}hz}{\cancel{2}}(2z+h) + z(z+h)(2h)\right] = \frac{1}{2h^2}\left[2hz^2 + \frac{2h^3}{3} - 4hz^2 - 2h^2z + 2hz^2 + 2h^2z\right] =$$

$$= \frac{2h^3}{3}\cdot\frac{1}{2h^2} = h/3$$

$$\bullet\ \left[W_1 : \int_a^b \frac{(x-a)(x-b)}{(h)(-h)}\,dx = -\frac{1}{h^2}\int_a^b (x^2-(a+b)x+ab)\,dx = -\frac{1}{h^2}\left[\frac{x^3}{3} - \frac{(a+b)x^2}{2} + abx\right]_a^b =\right.$$

$$= -\frac{1}{h^2}\left[2\cancel{h}z^2 + \frac{2h^3}{3} - \underbrace{2hz(2z)}_{4\cancel{h}z^2} + \underbrace{(z^2-h^2)\cdot 2h}_{(2\cancel{h}z^2-2h^3)}\right] = -\frac{1}{h^2}\left[\underbrace{\frac{2h^3}{3} - 2h^3}_{-4h^3/3}\right] = \boxed{\frac{4h}{3}}$$

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3.

$$\cdot\ W_2 = \int_a^b \frac{(x-a)(x-z)}{(2h)(h)}dx = \frac{1}{2h^2}\int_a^b \left(x^2-(a+z)x+az\right)dx = \frac{1}{2h^2}\left[\frac{x^3}{3}-\frac{(a+z)x^2}{2}+azx\right]_a^b =$$

$$= \frac{1}{2h^2}\left[2\cancel{h}z^2+\frac{2h^3}{3}-(-h)\cdot 2hz+\underbrace{(z^2-zh)2h}_{\cancel{2hz^2-2h^2z}}\right] = \frac{2h^3}{2h^2\cdot 3} = \boxed{h/3}$$

$$\boxed{Q = \frac{h}{3}\cdot f_0 + \frac{4h}{3}f_1 + \frac{h}{3}f_2 = \frac{h}{3}(f_0+4f_1+f_2)}$$

$\cdot$ $I = Q + E$

$$I = \int_{-h}^{h} x^3dx = \frac{h^4-h^4}{4} = 0 = \frac{h}{3}\cdot(h^3+\cancel{4\cdot 0}-h^3) + E \rightarrow E = 0$$

$$I = \int_{-h}^{h} x^4dx = \frac{2h^5}{5} = \frac{h}{3}(h^4+h^4) + E \longrightarrow E \neq 0$$

$\rightarrow E = K f^{(4)}(\xi) = 4\cdot3\cdot2\cdot1\ K = 24K$

$$\frac{2h^5}{5} - \frac{2h^5}{3} = 24K \rightarrow K = \frac{6h^5-10h^5}{24\cdot15} = \frac{-h^5}{90}$$

$$\boxed{E = -\frac{h^5}{90}\cdot f^{(4)}(\xi)}$$

b) Derive the compound simpson rule and trunc. error

$M = \frac{(b-a)}{2h}$; $x_i = a + (i-1)h$,

$$Q_C = \frac{h}{3}(f_1+4f_2+f_3) + \frac{h}{3}(f_3+4f_4+f_5) + \dots + \frac{h}{3}(f_{2M-1}+4f_{2M}+f_{2M+1}) =$$

$$= \boxed{\frac{h}{3}(f_1+4f_2+2f_3+4f_4+2f_5+\dots+2f_{2M-1}+4f_{2M}+f_{2M+1})}$$

$$E_c = \frac{\Sigma - h^5 f^{4)}(\xi)}{90} = \frac{M(-h^5)}{90} \frac{\Sigma f^{4)}(\xi_i)}{M} = -\frac{h^5 M}{90} \overline{f^{4)}}$$

$$\boxed{E_c = -\frac{h^5 \cdot (b-a)}{90 \cdot 2h} \overline{f^{4)}} = -\frac{h^4 (b-a) \overline{f^{4)}(\xi)}}{180}}$$

c) $\int_0^{0'8} f(x)\, dx$; COMPOUND simpson;

x_i	0	0'1	0'2	0'3	0'4	0'5	0'6	0'7	0'8
$f(x_i)$	0	2'122	3'0244	3'2568	3'1399	2'8579			

M = 4 subintervals

h = 0'1

$$\boxed{Q = \frac{h}{3} \cdot \left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + 2f_6 + 4f_7 + f_8 \right] =}$$

$$= \frac{0'1}{3} \left[\underbrace{(0 + 1'8358)}_{1'8358} + \underbrace{4(2'122 + 3'2568 + 2'8579 + 2'1639)}_{41'6024} + 2\underbrace{(3'0244 + 3'1399 + 2'5140)}_{17'3566} \right] = \boxed{2'0264733}$$

d) $\int_{0'5}^{1} e^{x^2} dx$; $E \leq 10^{-8} \longrightarrow M?$

UPPERBOUND

$$E_c = \left| -\frac{h^4 (b-a) \overline{f^{4)}}(\xi)}{180} \right| \leq \frac{h^4 \cdot 0'5}{180} \cdot \overline{f^{4)}(\xi)} \leq 10^{-8}$$

$$f = e^{x^2} \rightarrow f' = 2xe^{x^2} \rightarrow f'' = 2e^{x^2} + 4x^2 e^{x^2} \rightarrow f''' = 12xe^{x^2} + 8x^3 e^{x^2} \rightarrow f^{IV} = 12e^{x^2} + 48x^2 e^{x^2} + 16x^4 e^{x^2}$$

$$\hookrightarrow f^{4)}(1) = (12 + 48 + 16)e^1 = 206'589$$

$$h^4 \leq \frac{10^{-8} \cdot 180}{0'5 \cdot 206'589} = 17'42587 \cdot 10^{-9} \longrightarrow h \leq 0'011489 \longrightarrow M = \frac{(b-a)}{2h} = \frac{0'5}{2\ 0'011489} = 21'759$$

$$\hookrightarrow \boxed{M = 22}$$

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4. $\begin{cases} y'' - 2y = \cos(t) - e^t \\ y(0) = -1 \; ; \; y'(0) = 2 \end{cases}$ 2 steps of RK4 to estimate t=0'2

$$\boxed{Y_{k+1} = Y_k + \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}}$$

$$\begin{cases} K_1 = f(T_k, Y_k)\, h_k \\ K_2 = f(T_k + h_k/2, \; y_k + K_1/2)\, h_k \\ K_3 = f(t_k + h_k/2, \; y_k + K_2/2)\, h_k \\ K_3 = f(T_k + h_k, \; y_k + K_3)\, h_k \end{cases}$$

* DEFINING THE SYSTEM OF ODE'S:

$$\begin{cases} Y = Y_1 \\ Y' = Y_2 = Y'_1 \end{cases} \longrightarrow \begin{cases} Y'_1 = Y' = Y_2 \\ Y'_2 = y'' = 2Y + \cos(t) - e^t \end{cases}$$

$$\mathcal{Y} = \begin{pmatrix} Y \\ Y' \end{pmatrix} \; ; \quad f = \begin{pmatrix} Y' \\ 2Y + \cos(t) - e^t \end{pmatrix}$$

$$t = 0 \rightarrow \mathcal{Y} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

we want to reach t=0'2 in 2 STEPS ⟶ h=0'1

$\textcircled{k=0}$ $t = 0, \; \mathcal{Y} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

- $K_1 = 0'1 \cdot \begin{pmatrix} 2 \\ -2 + \cancel{\cos(0)} - \cancel{e^0} \end{pmatrix} = \begin{pmatrix} 0'2 \\ -0'2 \end{pmatrix}$

- $K_2 = 0'1 \cdot f\left(0'05, \begin{pmatrix} -0'9 \\ 1'9 \end{pmatrix}\right) = 0'1 \begin{pmatrix} 1'9 \\ -1'85252 \end{pmatrix} = \begin{pmatrix} 0'19 \\ -0'185252 \end{pmatrix}$

- $K_3 = 0'1\, f\left(0'05, \begin{pmatrix} -0'905 \\ 1'907374 \end{pmatrix}\right) = 0'1 \begin{pmatrix} 1'907374 \\ -1'862521 \end{pmatrix} = \begin{pmatrix} 0'1907374 \\ -0'186251 \end{pmatrix}$

- $K_4 = 0'1\, f\left(0'1, \begin{pmatrix} -0'80926 \\ 1'813749 \end{pmatrix}\right) = 0'1 \begin{pmatrix} 1'813745 \\ -1'728687 \end{pmatrix} = \begin{pmatrix} 0'1813749 \\ -0'1728687 \end{pmatrix}$

$$\boxed{\underset{\sim}{Y}(0'1) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1'1428497 \\ -1'1158747 \end{pmatrix} = \begin{pmatrix} -0'80952505 \\ 1'8140204 \end{pmatrix}}$$

$\textcircled{k=1}$ $t = 0'1$, $\underset{\sim}{Y}_1 = \begin{pmatrix} -0'80952505 \\ 1'8140204 \end{pmatrix}$

- $k_1 = 0'1 \cdot \begin{pmatrix} 1'8140204 \\ -1'7292168 \end{pmatrix} = \begin{pmatrix} 0'18140204 \\ -0'17292168 \end{pmatrix}$

- $k_2 = 0'1 \cdot f\left(0'15, \begin{pmatrix} -0'71880398 \\ 1'7275596 \end{pmatrix}\right) = \begin{pmatrix} 0'17275596 \\ -0'16106711 \end{pmatrix}$

- $k_3 = 0'1 \cdot f\left(0'15, \begin{pmatrix} -0'72314707 \\ 1'7334868 \end{pmatrix}\right) = \begin{pmatrix} 0'17334868 \\ 0'16193573 \end{pmatrix}$

- $k_4 = 0'1 \cdot f\left(0'2, \begin{pmatrix} -0'63617637 \\ 1'65208467 \end{pmatrix}\right) = \begin{pmatrix} 0'165208467 \\ -0'1513689 \end{pmatrix}$

$$\boxed{\underset{\sim}{Y}_2 = \underset{\sim}{y}(0'2) = \begin{pmatrix} -0'80952505 \\ 1'8140204 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1'038819787 \\ -0'97029626 \end{pmatrix} = \begin{pmatrix} -0'636388418 \\ 1'65230436 \end{pmatrix}}$$

$\hookrightarrow$ $\boxed{Y(0'2) \simeq -0'636388418}$

5 [10, b]

$$\begin{cases} y' = t^2 - ty \\ y(10) = 3 \end{cases}$$

$t \geq 10$ RK4 , $h = 0'2$; abs stability RK4 = $(-2'78, 0)$

Which interval is adequate in order to not have instabilities

$$\begin{cases} y = y \\ y' = t^2 - ty = f(t, y) \longrightarrow J = (-t) \end{cases}$$

$\hookrightarrow$ $(-t)y + \underbrace{t^2}_{\text{Kte}}$

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5. $\text{eigs}(Jh) = h \cdot \text{eigs}(J)$; $J = (-t) \rightarrow \text{eigs}(J) = -t$

- Abs. STABILITY REGION : $(-2'78, 0)$

$-2'78 < -t \cdot h < 0$ $\nearrow t > 0$ ✓ $\searrow t < \frac{2'78}{0'2} = 13'9 \rightarrow \boxed{t \in]0, 13'9]}$

T

6 Justify which of the following expressions can be truncation error terms of an interp. num. diff formula to approximate $f'(z)$. For the affirmative ones give some valid position of the nodes

$$E_1 = \frac{f^{3}(\xi)}{3!}\Pi(z) \ ; \ E_2 = \frac{f^{4)}(\xi)}{4!}\Pi(z) \ ; \ E_3 = \frac{f^{3)}(\xi)}{3!}\Pi'(z) \ ; \ E_4 = \frac{f^{3)}(\xi)}{3!}\Pi(z) + \frac{f^{4)}(\eta)}{4!}\Pi'(z)$$

$$E_5 = \frac{f^{3)}(\xi)}{3!}\Pi'(z) + \frac{f^{4)}(\eta)}{4!}\Pi(z)$$

$$\boxed{E} = \left(f[x_0, x_1, x_2, \dots, x_n, z]\,\Pi(z)\right)' = f'[x_0, x_1, x_2, \dots x_n, z]\cdot\Pi(z) + f[x_0, x_1, x_2, \dots, x_n, z]\,\Pi'(z) =$$

$$= f[x_0, x_1, x_2, \dots, x_n, z, z]\,\Pi(z) + f[x_0, x_1, x_2, \dots, x_n, z]\,\Pi'(z) =$$

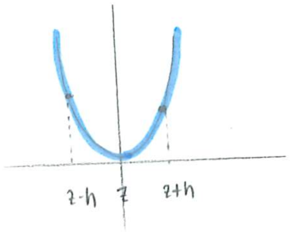
$$= \frac{f^{n+k_1+1)}(\xi_1)}{(n+k_1+1)!}\Pi(z) + \frac{f^{n+k_2+1)}(\xi_2)}{(n+k_2+1)!}\Pi'(z) = \boxed{\frac{f^{n+2)}(\xi_1)}{(n+2)!}\Pi(z) + \frac{f^{n+1)}(\xi_2)}{(n+1)!}\Pi'(z)}$$

- $\underline{E_1} = \frac{f^{3)}(\xi)}{3!}\Pi(z)$ $\rightarrow n = 1 \rightarrow nn = 2$

$\rightarrow \Pi'(z)\cdot \cancel{f^{2)}} = 0 \rightarrow \Pi'(z) = 0 \rightarrow$ EVEN NUMBER OF NODES WITH CENTRAL SUMMERY $\rightarrow$ PAR $\rightarrow nn = 2$ ✓

impossible

↳ IT IS POSSIBLE → PARABOLA:



EVEN $nn \Longrightarrow \Pi'(z) = 0$
(only in one direction)

(SUFFICIENT)

- $E_2 = \frac{f^{4)}(\xi)}{4!}\pi(z)$ $\nearrow$ $n=2 \rightarrow nn=3$
  $\searrow$ $\pi'(z)\,\cancel{f^{3)}}(\xi)=0 \rightarrow \pi'(z)=0$ (impossible)

$\hookrightarrow$ EVENTHOUG $nn \equiv$ ODD we can choose 3 node: x_0, x_1, x_2 and then, define z as the rel. max or min such that the other 2 nodes will be defined by z and the stepsize h $\rightarrow$ POSSIBLE

$\hookrightarrow x_i = z \pm k_i h$

$\pi(z)=0 \Longleftrightarrow z$ is a node

Both dir.

NECESSARY AND SUFF.

- $E_3 = \frac{f^{3)}(\xi)}{3!}\pi'(z)$ $\nearrow$ $n=2 \rightarrow nn=3$
  $\searrow$ $f^{4)}(\eta)\pi(z)=0$ $\nearrow f^{4)}(\eta)=0$, $\searrow \pi(z)=0$ $\longrightarrow$ POSSIBLE

$\hookrightarrow$ Ex: $x_0 = z$, $x_1 = h + z k_1$, $x_2 = h + z k_2$

- $E_4 = \frac{f^{3)}(\xi)}{3!}\pi(z) + \frac{f^{4)}(\eta)}{4!}\pi'(z) \longrightarrow$ IMPOSSIBLE

- $E_5 = \frac{f^{4)}(\xi)}{4!}\pi(z) + \frac{f^{3)}(\eta)}{3!}\pi'(z) \longrightarrow n=2 \rightarrow nn=3$ (EVEN)

$\hookrightarrow$ IF z is not a node $\rightarrow$ POSSIBLE $\rightarrow$ Ex: $x_0 = z+h$, $x_1 = z+2h$, $x_2 = z+3h$

7. Find an interpolatory num. diff formula (without its trunc. error) to estimate $f''(z)$ in terms of the value of f at $z, z+2h, z+3h$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2) =$$

$$= \underbrace{\frac{(x-z-2h)(x-z-3h)}{(-2h)(-3h)}}_{L_0(x)}f(z) + \underbrace{\frac{(x-z)(x-z-3h)}{(2h)(-h)}}_{L_1(x)}f(z+2h) + \underbrace{\frac{(x-z)(x-z-2h)}{(3h)(h)}}_{L_2(x)}f(z+3h)$$

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$$\cdot\ L_0(x) = \frac{(x-z-2h)(x-z-3h)}{6h^2} \longrightarrow L_0'(x) = \frac{(x-z-2h)+(x-z-3h)}{6h^2} \longrightarrow L_0''(x) = \frac{2}{6h^2} = \frac{1}{3h^2}$$

$$\cdot\ L_1(x) = \frac{(x-z)(x-z-3h)}{-2h^2} \longrightarrow L_1'(x) = \frac{(x-z)+(x-z-3h)}{-2h^2} \longrightarrow L_1''(x) = -\frac{1}{h^2}$$

$$\cdot\ L_2(x) = \frac{(x-z)(x-z-2h)}{(3h)(h)} \longrightarrow L_2'(x) = \frac{(x-z)+(x-z-2h)}{3h^2} \longrightarrow L_2''(x) = \frac{2}{3h^2}$$

$$\boxed{D = f''(z) =} \frac{f(x_0)}{3h^2} - \frac{f(x_1)}{h^2} + \frac{2f(x_2)}{3h^2} = \boxed{\frac{f(z) - 3f(z+2h) + 2f(z+3h)}{3h^2}}$$

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$$\xrightarrow{\quad} \left\{\begin{array}{l} P(1) \\ P(-1) \\ P'(2) \end{array}\right. \text{ not GIVEN } \longrightarrow \text{ not an osculating pol.}$$

$P(-1) = f(-1)$; $P'(-1) = f'(-1)$; $P'(1) = f'(1)$; $P''(2) = f''(2)$

Is a polynomial uniquely determined from the data above?

$\longrightarrow$ 4 conditions

- $p(x) = a + bx + cx^2 + dx^3$

$\hookrightarrow p(-1) = a - b + c - d$

- $p''(x) = 2c + 6dx$

$\hookrightarrow p''(2) = 2c + 12d$

- $p'(x) = b + 2cx + 3dx^2$

$\hookrightarrow p'(-1) = b - 2c + 3d$

$\hookrightarrow p'(1) = b + 2c + 3d$

$$\left\{\begin{array}{l} a - b + c - d = f(-1) \\ b - 2c + 3d = f'(-1) \\ b + 2c + 3d = f(1) \\ 2c + 12d = f''(2) \end{array}\right.$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & f(-1) \\ 0 & 1 & -2 & 3 & f'(-1) \\ 0 & 1 & 2 & 3 & f'(1) \\ 0 & 0 & 2 & 12 & f''(2) \end{array}\right) = \langle R_3 - R_2 \rangle = \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & f(-1) \\ 0 & 1 & -2 & 3 & f'(-1) \\ 0 & 0 & 4 & 0 & f'(1) - f'(-1) \\ 0 & 0 & 2 & 12 & f''(2) \end{array}\right) = \langle R_4 - R_3/2 \rangle$$

$$= \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & f(-1) \\ 0 & 1 & -2 & 3 & f'(-1) \\ 0 & 0 & 4 & 0 & f'(1) - f'(-1) \\ 0 & 0 & 0 & 12 & f''(2) - \frac{(f'(1) - f'(-1))}{2} \end{array}\right) \longrightarrow \text{USING BACKWARD SUBSTITUTION WE CAN SOLVE IT}$$

$\text{Rank}(A) = 4 = \text{Rank}(A/b) \longrightarrow$ DETERMINED SYSTEM

$\exists$ a UNIQUE POLYNOMIAL OF DEGREE ≤ 3 SATISFYING THOSE 4 conditions

2.

A)

x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$
-1	-1	4	-6
1	3	4	

$x = 0'5 \rightarrow$ optimally evaluate at that point $\rightarrow$ HORNER-LIKE ALGORITHM

x_i	$h_{i,0}$	$h_{i,1}$	h_{i2}	h_{i3}	h_{i4}	h_{i5}
-1	-1	—	—	—	—	-
-1	-1	4	—	-	-	-
-1	-1	4	-3	-	-	-
1	3	2	-1	1	-	-
1	3	4	1	1	0	-
1	3	4	4	1'5	0'25	0'125

$$P_3(x) = -1 + 4(x+1) - 3(x+1)^2 + 1(x+1)^3$$

$$P_3(x) = \underbrace{-1 + \underbrace{(x+1)\,\underbrace{\left[4 + \underbrace{(x+1)\,\underbrace{(-3+\underbrace{(x+1)}_{1'5})}_{-1'5}}_{-2'25}\right]}_{1'75}}_{2'625}}_{1'625}$$

$x = 0'5$

$\longrightarrow \boxed{P_3(0'5) = 1'625}$

B) Knowing that $f''(1) = 8$ estimate the error made in the previous section

$$e(x) = P_5(x) - P_3(x)$$

$$P_5(x) = -1 + 4(x+1) - 3(x+1)^2 + (x+1)^3 + 0\,\overset{0}{\cancel{(x+1)^3(x-1)}} + 0'125\,(x+1)^3(x-1)^2$$

$$e(x) = 0'625(x+1)^3(x-1)^2 \longrightarrow \boxed{e(0'5)} = 0'125(1'5)^3(-0'5)^2 = \boxed{0'10546875}$$

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3. Which of the following matrices could be the coeff matrix T of the system to be solved to calculate a nat. CUBIC SPLINE with 5 nodes?

$$A_1 = \begin{pmatrix} 0'4 & 0'2 & 0 \\ 0'2 & 0'5 & 0'3 \\ 0 & 0'3 & 0'6 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0'6 & 0'2 & 0 \\ 0'2 & 0'8 & 0'2 \\ 0 & 0'2 & 0'6 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0'2 & 0'1 & 0 & 0 & 0 \\ 0'1 & 0'6 & 0'2 & 0 & 0 \\ 0 & 0'2 & 0'6 & 0'4 & 0 \\ 0 & 0 & 0'1 & 0'6 & 0'2 \\ 0 & 0 & 0 & 0'2 & 0'4 \end{pmatrix}$$

$A_1, A_2, A_3 \longrightarrow$ TRIDIAGONAL SIMMETRIC MATRICES

$$T = \begin{pmatrix} 2(h_0+h_1) & h_1 & 0 & \cdots & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 \\ 0 & \vdots & \ddots & & \vdots \\ \vdots & & \vdots & \ddots & \\ 0 & 0 & \cdots & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix}$$

→ not necessary rn.

- 5 nodes → 3 UNKNOWNS because we already know: $\underbrace{P_0''(x) = 0 = P_{n-1}''(x)}_{1\text{ST and last nodes}} \longrightarrow \cancel{A_3}$

- THEY also have to be STRICTLY DIAG. DOMINAT:

A_1 : $0'4 > 0'2$; $\underbrace{0'5 = 0'5}_{\text{NOT GREATER}}$ ⓧ

* DUDO PORQUE NO SE CUMPLE LO DE LA MATRIZ DE ARRIBA

A_2 : $0'6 > 0'2$; $0'8 > 0'4$, $0'6 > 0'2$ ✓ $\longrightarrow$ $\boxed{A_2 = \begin{pmatrix} 0'6 & 0'2 & 0 \\ 0'2 & 0'8 & 0'2 \\ 0 & 0'2 & 0'6 \end{pmatrix}}$

4 calculate the EXACT VALUE of:

$$\int_1^2 \left[(x-1)^2 + \frac{x^3}{\sqrt{(x-1)(2-x)}} \right] dx$$ using num. quadrature rules

$$I_1 = \int_1^2 (x-1)^2\, dx \qquad I_2 = \int_1^2 \frac{x^3}{\sqrt{(x-1)(2-x)}}\, dx$$

(I_2) $x = 1'5 + 0'5t$ $\quad dx = 0'5\,dt$ $\longrightarrow$ Int. LIMITS $\left\{ \begin{array}{l} t = (2-1'5)/0'5 = 1 \\ t = (1-1'5)/0'5 = -1 \end{array} \right.$

$$I_2 = \int_1^2 \frac{x^3}{\sqrt{(x-1)(2-x)}}\, dx = \int_{-1}^{1} \frac{(1'5+0'5t)^3 \cdot 0'5}{\sqrt{(0'5+0'5t)(0'5-0'5t)}}\, dt = \int_{-1}^{1} \frac{(1'5+0'5t)^3 \cdot \cancel{0'5}}{\cancel{0'5}\sqrt{(1+t)(1-t)}}\, dt = \underbrace{\int_{-1}^{1} \frac{(1'5+0'5t)^3}{\sqrt{1-t^2}}\, dt}_{\boxed{\text{GAUSS-CHEBYSHEV}}}$$

$g(t) = (1'5+0'5t)^3$

$$\left\{ \begin{array}{l} W = \dfrac{\pi}{n+1} \\ t_i = \cos\left(\dfrac{\pi/2 + i\pi}{n+1}\right) \end{array} \right. \longrightarrow$$ I am working with 2 nodes ($n=1$)

$\hookrightarrow W = \pi/2$

$\hookrightarrow t_i = \cos\left(\dfrac{\pi/2 + i\pi}{2}\right)$ $\nearrow t_0 = \sqrt{2}/2$ $\searrow t_1 = -\sqrt{2}/2$

$$\boxed{I_2 = \frac{\pi}{2}\left[\underbrace{\underbrace{(1'5+0'5\sqrt{2}/2)^3}_{6'36817956} + \underbrace{(1'5-0'5\sqrt{2}/2)^3}_{1'50682044}}_{7'875} \right] = 12'37002107}$$

(I_1) $\displaystyle\int_1^2 (x-1)^2\, dx = \int_1^2 (x^2-2x+1)\, dx$ $\longrightarrow$ $n=2$ $\longrightarrow$ SIMPSON RULE

$$I_1 = \int_1^2 (x-1)^2\, dx = \frac{h}{3}(f_0 + 3f_1 + f_3)$$

$\left\{ \begin{array}{l} x_0 = 1 \\ x_1 = 1'5 \\ x_2 = 2 \end{array} \right.$ $h = 0'5$

$\hookrightarrow$ $\boxed{I_1 = \dfrac{0'5}{3}\left(0 + 4 \cdot 0'5^2 + 1\right) = \dfrac{1}{3} = 0'3\overset{\frown}{3}}$

$$\boxed{I = J_1 + J_2 = 12'70335441}$$

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6. $\begin{cases} y''+ty'+y=0 \rightarrow y''=-(ty'-y) \\ Y(0)=1\ ,\ y'(0)=2 \end{cases}$ ∴ A) Take 1 step of RK4 to estimate $y(0'2)$, $y'(0'2)$

$$\underline{Y}=\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}=\begin{pmatrix} Y \\ Y' \end{pmatrix} \longrightarrow Y'=\begin{pmatrix} Y'_1 \\ Y'_2 \end{pmatrix}=\begin{pmatrix} \overset{y_2}{Y'} \\ Y'' \end{pmatrix}=\begin{pmatrix} Y' \\ -(tY'+Y) \end{pmatrix}=f(t_k,Y_k) \quad ; \quad h_k=0'2$$

RK4 :

$$\begin{aligned} k_1 &= f(t_k, Y_k)\, h_k \\ k_2 &= f(t_k+h_k/2,\ Y_k+k_1/2)\, h_k \\ k_3 &= f(t_k+h_k/2,\ Y_k+k_2/2)\, h_k \\ k_4 &= f(t_k+h_k,\ Y_k+k_3)\, h_k \end{aligned}$$

$$\boxed{Y_{k+1}=Y_k+\frac{k_1+2k_2+2k_3+k_4}{6}}$$

(K=0) $Y_0=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $t_0=0$

$$k_1=0'2\cdot\begin{pmatrix} 2 \\ -1 \end{pmatrix}=\begin{pmatrix} 0'4 \\ -0'2 \end{pmatrix}$$

$$k_2=0'2\, f\left(0'1,\begin{pmatrix} 1'2 \\ 1'9 \end{pmatrix}\right)=0'2\begin{pmatrix} 1'8 \\ -1'39 \end{pmatrix}=\begin{pmatrix} 0'38 \\ -0'278 \end{pmatrix}$$

$$k_3=0'2\, f\left(0'1,\begin{pmatrix} 1'19 \\ 1'861 \end{pmatrix}\right)=0'2\begin{pmatrix} -1'861 \\ -1'3761 \end{pmatrix}=\begin{pmatrix} 0'3722 \\ -0'27522 \end{pmatrix}$$

$$k_4=0'2\, f\left(0'2,\begin{pmatrix} 1'3722 \\ 1'72478 \end{pmatrix}\right)=0'2\begin{pmatrix} 1'72478 \\ 1'717156 \end{pmatrix}=\begin{pmatrix} 0'344956\ , \\ -0'3434312 \end{pmatrix}$$

$$\underline{Y}(0'2)=\begin{pmatrix} 1 \\ 2 \end{pmatrix}+\frac{1}{6}\begin{pmatrix} 2'249\cdot356 \\ -1'6498712 \end{pmatrix}=\boxed{\begin{pmatrix} 1'374892667 \\ 1'725021467 \end{pmatrix}=\begin{pmatrix} Y(0'2) \\ Y'(0'2) \end{pmatrix}}$$

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B)

(0)	0	1	2
	t_i	$Y(t_i)$	$Y'(t_i)$
(1)	0'2	1'37489	1'72502
(2)	0'4	1'68176	1'3273
(3)	0'6	1'9011	0'85934

$$\underset{\sim}{f}(t,y) = \begin{pmatrix} y' \\ -(y+y't) \end{pmatrix} \; ; \; \underset{\sim}{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$h = 0'2$

$\boxed{n=3}$

Estimate $y(0'8)$ and $y'(0'8)$ using Adam's predictor-corrector method with $Y.0'1$ precision

$$Y_{n+1} = Y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right] \qquad n = 3, 4, \dots, N-1$$

$$Y_{n+1} = Y_n + \frac{h}{24}\left[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right] \qquad n = 2, 3, \dots, N-1$$

- $$Y_{4P} = \begin{pmatrix} Y(0'6) \\ Y'(0'6) \end{pmatrix} + \frac{0'2}{24}\left[55 \cdot \overset{(3)}{f}(0'6, \underset{\sim}{y}(0'6)) - 59 \overset{(3)}{f}(0'4, \underset{\sim}{y}(0'4)) + 37 \overset{(2)}{f}(0'2, \underset{\sim}{y}(0'2)) - 9 \overset{(1)}{f}(0, \overset{(0)}{\underset{\sim}{y}(0)})\right]$$

$$\boxed{Y_{4P}} = \begin{pmatrix} 1'9011 \\ 0'85934 \end{pmatrix} + \frac{1}{120}\left[55\begin{pmatrix} 0'85934 \\ -2'416704 \end{pmatrix} - 59\begin{pmatrix} 1'3273 \\ -2'21268 \end{pmatrix} + 37\begin{pmatrix} 1'72502 \\ -1'719894 \end{pmatrix} - 9\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right] = \boxed{\begin{pmatrix} 2'024256169 \\ 0'384428435 \end{pmatrix}}$$

- $$Y_{4C1} = \begin{pmatrix} Y(0'6) \\ Y'(0'6) \end{pmatrix} + \frac{0'2}{24}\left[9 \cdot f(0'8, \underset{\sim}{Y_{4P1}}) + 19 \cdot f(0'6, \underset{\sim}{y}(0'6)) - 5 f(0'4\ \underset{\sim}{y}(0'4)) + f(0'2, \underset{\sim}{y}(0'2))\right]$$

$$\boxed{Y_{4C1}} = \begin{pmatrix} 1'9011 \\ 0'85934 \end{pmatrix} + \frac{1}{120}\left[9\begin{pmatrix} 0'38428435 \\ -2'331683647 \end{pmatrix} + 19\begin{pmatrix} 0'85934 \\ -2'416704 \end{pmatrix} - 5\begin{pmatrix} 1'3273 \\ -2'21268 \end{pmatrix} + \begin{pmatrix} 1'72502 \\ -1'719894 \end{pmatrix}\right] = \boxed{\begin{pmatrix} 2'025054493 \\ 0'379681476 \end{pmatrix}}$$

- PRECISION : $\boxed{|(Y_{4C1} - Y_{4P})| \cdot 100 / Y_{4C1} \leq 0'1}$

$$\begin{pmatrix} 0'03942245 \\ 1'212298806 \end{pmatrix} = \begin{pmatrix} < 0'1 \\ > 0'1 \end{pmatrix}$$

- $$\boxed{Y_{4C2}} = \begin{pmatrix} 1'996233167 \\ 0'55455775 \end{pmatrix} + \frac{9}{120}\begin{pmatrix} 0'379681476 \\ -2'328799674 \end{pmatrix} = \boxed{\begin{pmatrix} 2'024709278 \\ 0'379897774 \end{pmatrix}}$$

- PRECISION : $$\frac{|(Y_{4C2} - Y_{4C1})| 100}{Y_{4C2}} = \begin{pmatrix} 0'01705531 \\ 0'056935841 \end{pmatrix} = \begin{pmatrix} < 0'1 \\ < 0'1 \end{pmatrix}$$ ✓

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8 T A) $f''(z)$ using base functions; $\overset{x_0}{f(z-2h)}$, $\overset{x_1}{f(z-h)}$, $\overset{x_2}{f(z)}$

- $L_0(x) = \dfrac{(x-z+h)(x-z)}{(-h)(-2h)} \longrightarrow L_0'(x) = \dfrac{(x-z+h)+(x-z)}{2h^2} \longrightarrow \boxed{L_0''(x) = \dfrac{1}{h^2} = L''_0(z)}$

- $L_1(x) = \dfrac{(x-z+2h)(x-z)}{(h)(-h)} \longrightarrow L_1'(x) = \dfrac{(x-z+2h)+(x-z)}{-h^2} \longrightarrow \boxed{L_1''(x) = \dfrac{-2}{h^2} = L''_1(z)}$

- $L_2(x) = \dfrac{(x-z+2h)(x-z+h)}{(2h)(h)} \longrightarrow L_2'(x) = \dfrac{(x-z+2h)+(x-z+h)}{2h^2} \longrightarrow \boxed{L_2''(x) = \dfrac{1}{h^2} = L''_2(z)}$

$$\boxed{f''(z) = \frac{1}{h^2}\cdot f(z-2h) - \frac{2}{h^2} f(z-h) + \frac{1}{h^2} f(z) + E = \frac{f(z-2h) - 2f(z-h) + f(z)}{h^2} + E}$$

B) Derive the E (num. error) for a interp. diff. formula.

$$\boxed{E =} \left(f[x_0, x_1, \ldots, x_n, z]\,\Pi(z)\right)'' = \left(f'[x_0, x_1, \ldots, x_n, z]\,\Pi(z) + f[x_0, x_1, x_2, \ldots, x_n, z]\,\Pi'(z)\right)' =$$

$$= f''[x_0, x_1, \ldots, x_n, z]\cdot\Pi(z) + 2f'[x_0, x_1, \ldots, x_n, z]\cdot\Pi'(z) + f[x_0, x_1, \ldots, x_n, z]\cdot\Pi''(z) =$$

$$= f[x_0, x_1, \ldots, x_n, z, \underbrace{z, z}_{k=2}]\,\Pi(z) + 2f[x_0, x_1, \ldots, x_n, z, \underbrace{z}_{k=1}]\,\Pi'(z) + f[x_0, x_1, \ldots, x_n, \underbrace{z}_{k=0}]\,\Pi''(z) =$$

$$\boxed{= \frac{f^{n+3)}(\xi_1)}{(n+3)!}\cdot\Pi(z) + 2\frac{f^{n+2)}(\xi_2)}{(n+2)!}\,\Pi'(z) + \frac{f^{n+1)}(\xi_3)}{(n+3)!}\,\Pi''(z)}$$

c) Obtain the error for A)

$$\begin{cases} \Pi(x) = (x-z+2h)(x-z+h)(x-z) \longrightarrow \Pi(z) = (2h)(h)(0) = 0 \\ \Pi'(x) = (x-z+2h)(x-z+h) + (x-z+2h)(x-z) + (x-z+h)(x-z) \longrightarrow \Pi'(z) = 2h^2 \\ \Pi''(x) = (x-z+2h) + (x-z+h) + (x-z+2h) + (x-z) + (x-z+h) + (x-z) \rightarrow \Pi''(z) = 4h + 2h = 6h \end{cases}$$

$$E = 2 \cdot \frac{f^{n+2)}(\xi_1)}{(n+2)!} \cdot 2h^2 + \frac{f^{n+1)}(\xi_2)}{(n+1)!} 6h \longrightarrow \boxed{n=2} \rightarrow E = 2 \cdot \frac{f^{4)}(\xi_1)}{4!} 2h^2 + \frac{f^{3)}(\xi_2)}{3!} 6h$$

$$\hookrightarrow \boxed{E = \frac{f^{4)}(\xi_1) h^2}{6} + f^{3)}(\xi_2) h \quad \text{f.s. } \xi_1, \xi_2 \in (z-2h, z)}$$

2016. EXTRA

1. * working with 5 significant digits

A) $f(x) = Ln(x)$ interval: $[2,3] \rightarrow 11$ possible nodes considered to interpolate the function

$h = 0'1$; calculate $P_a(x)$ of degree ≤ 3 you' expect to minimize the abs. max. error

at $\boxed{x = 2'55}$

↳ AT LEAST 4 nodes ∈ TO THE INTERVAL

$x_0 = 2$, $x_1 = 2'1$, $x_2 = 2'2$, $x_3 = 2'3$, $\boxed{x_4 = 2'4,\ x_5 = 2'5,\ x_6 = 2'6,\ x_7 = 2'7}$ $x_8 = 2'8$, $x_9 = 2'9$, $x_{10} = 3$

↳ CLOSEST TO $x = 2'55$

x_i	$\Delta f_{i,0}$	Δf_{i1}	Δf_{i2}	Δf_{i3}	Δf_{i4}
2'4	0'87547	—	—	—	—
2'5	0'91629	0'04082	—	—	—
2'6	0'95551	0'03922	-0'0016	—	—
2'7	0'99325	0'03774	-0'00148	0'00012	—
2'8	1'0296	0'03635	-0'00139	0'00009	-0'00003

$h = 0'1 \rightarrow$ FINITE DIFFERENCES

*AS WE ARE USING SOME NODES FROM THE MIDDLE OF THE INTERVAL, WE MUST CARRY OUT A CHANGE OF VARIABLE SUCH THAT $x = x_0 + ht$

$$x = 2'4 + 0'1t \longrightarrow t = \frac{x - 2'4}{0'1} = 10x - 24$$

$$\boxed{q_3(t)} = \frac{0'87547}{0!} + \frac{0'04082\,t}{1!} - \frac{0'0016\,t(t-1)}{2!} + \frac{0'00012\,t(t-1)(t-2)}{3!} =$$

$$\boxed{= 0'87547 + 0'04082\,t - 0'0008\,t(t-1) + 0'00002\,t(t-1)(t-2)}$$

$$\hookrightarrow \boxed{P_3(t) = 0'87547 + 0'04082(10x-24) - 0'0008\,(10x-24)(10x-25) + 0'00002(10x-24)(10x-25)(10x-26)}$$

↳ HORNER - LIKE ALGORITHM

B) optimally calculate the value for $X=2'55$

$t= 10\cdot 2'55-24 = 1'5$

$$q_3(t)=\underbrace{0'87547+\underbrace{t\Big[\underbrace{0'04082+\underbrace{(t-1)\underbrace{\big(-0'0008+\underbrace{0'00002(t-2)}_{-0'00001}\big)}_{-0'00081}}_{-0'000405}}_{0'040415}\Big]}_{0'060623}}_{0'93609} \qquad t=1'5$$

$$\boxed{q_3(1'5) = p_3(2'55) = 0'93609}$$

C) Estimate the trunc. error. at $X=2'55$ using $f(2'8)$

$$e(t) = q_4(t) - q_3(t) = \Delta f_{414}\cdot\frac{t(t-1)(t-2)(t-3)}{4!} =$$

$$\boxed{e(1'5) = \frac{-0'00003 : 1'5(0'5)(-0'5)(-1'5)}{24} = -0'70313\cdot 10^{-6}}$$

D) Find the upperbound of the abs. - value truncation error of $P_3(x)$ $\forall x \in [2,3]$

$$E = \frac{f^{n+1)}(\xi)}{(n+1)!}\pi(x) \longrightarrow \boxed{n=3} \longrightarrow E = \frac{f^{4)}(\xi)}{4!}(x-2'4)(x-2'5)(x-2'6)(x-2'7)$$

$$f(x) = Ln(x) \to f'(x) = 1/x \to f'' = -1/x^2 \to f''' = 2/x^3 \to f^{4)} = -6/x^4$$

$$|E| = \left|\frac{-6/x^4}{24}\cdot(x-2'4)(x-2'5)(x-2'6)(x-2'7)\right| = \left|\frac{(x-2'4)(x-2'5)(x-2'6)(x-2'7)}{4x^4}\right|$$

* we have to choose between $x=2$ and $x=3$ → $x=2$ further from the points

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1. D) * However, we could also replace both and see which is the highest value

$$\boxed{|E_a(2)| =} \left| \frac{(-0'4)(-0'5)(-0'6)(-0'7)}{4 \cdot 2^4} \right| = \boxed{0'0013125}$$

↳ comprobation (x = 3) → $E(3) = \dfrac{(0'6)(0'5)(0'4)(0'3)}{4 \cdot 3^4} = 0'000111 < E(2)$

E) Calculate the interp. points for $P_3(x)$ of degree ≤ 3 → 4 nodes (nn) that interpolates f(x) minimizing (approximately) the max. abs-value over the interval [2,3]

CHEBYSHEV

$$\boxed{\cos((n+1)\theta_i) = 0} \rightarrow \cos(4\theta_i) = 0 \longrightarrow 4\theta_i = \frac{\pi}{2} + \pi k \longrightarrow t_i = \cos\left(\frac{\pi/2 + \pi k}{4}\right)$$

→ $t_i = \cos\left(\dfrac{\pi/2 + \pi k}{n+1}\right)$

$k=0 \longrightarrow t_0 = \cos(\pi/8) = 0'92388$

$k=1 \longrightarrow t_1 = \cos(3\pi/8) = 0'38268$

$k=2 \longrightarrow t_2 = \cos(5\pi/8) = -0'38268$

$k=3 \longrightarrow t_3 = \cos(7\pi/8) = -0'92388$

$t \in [-1,1]$

* CHANGE OF VARIABLE: $x = \dfrac{(3+2)}{2} + \dfrac{(3-2)}{2} t = 2'5 - 0'5t$

$X_0 = 2'3806$

$X_1 = 2'3087$

$X_2 = 2'6913$

$X_3 = 2'9619$

→

$f_0 = 0'86735$

$f_1 = 0'83668$

$f_2 = 0'99002$

$f_3 = 1'0858$

F) knowing that $P_b(2'55) = 0'93614$, compare the errors of $P_a(x)$ and $P_b(x)$ at 2'55

conclusion!

$$f(2'55) = Ln(2'55) = 0'93609$$

$$\left\{\begin{array}{l} e_b(2'55) = \text{exact} - \text{approx} = 0'93609 - 0'93614 = -5 \cdot 10^{-5} \\ \\ e_a(2'55) = -7'0313 \cdot 10^{-5} \end{array}\right\}$$

Both are errors in defect

$|e_b| < |e_a|$ — P_b will be better

G) upperbound error for P_b in [2, 3]

$$|E_b| = \left| \frac{-6/x^4}{24} (x - 2'3806)(x - 2'3087)(x - 2'6913)(x - 2'9619) \right|$$

$$\hookrightarrow \textcircled{x=2} \rightarrow \boxed{E_b(2) = \frac{(-0'3806)(-0'3087)(-0'6913)(-0'9619)}{4 \cdot 2^4} = 0'0012222} < |E_a(2)|$$

2. $S = \int_a^b \int_c^d f(x, y)\,dx\,dy$ 3 dec. digits. 1'6m x 1'2m $\left\{ \begin{array}{l} c = 0,\ d = 1'6 \\ a = 0,\ b = 1'2 \end{array} \right.$

{ Trapezoidal rule for one of the var
2nd Simpson for the other one

$h_y = 0'2$

$0'4 = h_x$

X \ y	0	0'2	0'4	0'6	0'8	1	1'2
0	0	0	0	0	0	0	0
0'4	0	3'123	4'794	5'319	4'794	3'123	0
0'8	0	3'818	5'960	6'647	5'960	3'818	0
1'2	0	3'123	4'794	5'319	4'794	3'123	0
1'6	0	0	0	0	0	0	0

SIMMETRY IN BOTH DIRECTIONS!

* 2nd Simpson: $\int_{u_0}^{u_3} f(u)\,du = \frac{3h}{8} [f(u_0) + 3f(u_1) + 3f(u_2) + f(u_3)] - \frac{3h^{5)}}{80} f^{4)}(\xi)$

* TRAPEZOIDAL RULE: $I = \frac{h}{2} [f_0 + f_1] - \frac{h^3 f^{2)}(\xi)}{12}$

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2.

2D.

$$\int_{0}^{1'2}\int_{0}^{1'6} f(x,y)\, dx\, dy$$

In ⓧ we will use THE TRAPEZOIDAL RULE and in ⓨ THE 2ND SIMPSON RULE

ⓧ $Q_{x,c} = \frac{h_x}{2}\left[f_0 + f_1\right] + \frac{h_x}{2}\left[f_1 + f_2\right] = \frac{h_x}{2}\left[f_0 + 2f_1 + f_2\right]$

$w_{0x} = 0'2$; $w_{1x} = 0'4$; $w_{2x} = 0'2$

ⓨ $Q_y = \frac{3h_y}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right]$

$w_{0y} = 0'075$; $w_{1y} = 0'225$; $w_{2y} = 0'225$; $w_{3y} = 0'075$

* we are creating a matrix : $\boxed{W_{ij} = \underset{\sim}{W_{xi}} \cdot \underset{\sim}{W_{yj}}'}$

$$W_{ij} = \begin{pmatrix} 0'2 \\ 0'4 \\ 0'2 \end{pmatrix} \begin{pmatrix} 0'075 & 0'225 & 0'225 & 0'075 \end{pmatrix} = \begin{pmatrix} 0'015 & 0'045 & 0'045 & 0'015 \\ 0'03 & 0'09 & 0'09 & 0'03 \\ 0'015 & 0'045 & 0'045 & 0'015 \end{pmatrix}$$

we are using the mentioned symmetry so f_{ij}:

$$f_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3'323 & 4'794 & 5'319 \\ 0 & 3'818 & 5'960 & 6'647 \end{pmatrix} \longrightarrow \boxed{q_{ij} = W_{ij} \cdot f_{ij}}$$

↳ no es una multiplicación de matrices

↳ es de elementos

elementwise product

!!

$$q_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0'281 & 0'431 & 0'159 \\ 0 & 0'171 & 0'268 & 0'099 \end{pmatrix}$$

↳ 1ST QUARTER ONLY ⟶ $Q = \text{sum}(\text{sum}(q_{ij})) = 1'409$

suma de todo

$Q = 4 \cdot 1'409 = 5'636 \longrightarrow \boxed{S \simeq 5'636}$

3. $y'(t) = -k\sqrt{y(t)}$

$y(0) = 3$; $k = 0'06$ (kte)

A) $y(0'5)$ using the Midpoint Method (modified Euler) with $h = 0'25$

MIDPOINT: $\boxed{y_{k+1} = y_k + k_2}$

$k_1 = f(t_k, y_k) \cdot h_k$

$k_2 = f(t_k + h_k/2,\ y_k + k_1/2) \cdot h_k$

} como los 2 primeros pasos del RK 4

$f(t, y) = -0'06\sqrt{y(t)}$

$h = 0'25$

$\bigcirc\!\!\!\!\!\!\;k=0$ $t = 0$, $y_0 = 3$

- $k_1 = 0'25 \cdot (-0'06\sqrt{3}) = -0'025980762$

- $k_2 = 0'25\, f(0'125,\ 2'987009619) = -0'025924451$

$y(0'25) = 3 - 0'025924451 = 2'974075549$,

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3 A)

(k=1) $t = 0'25$, $Y_1 = 2'974075549$

$k_1 = 0'25(-0'06\sqrt{2'974075549}) = -0'025868262$

$\rightarrow \boxed{Y(0'5) = 2'974188049}$

$k_2 = 0'25 \cdot f(0'4, 2'961141418) = -0'025811951$

B) $t = 30$: $\begin{cases} 0'6831 ; & h = 0'5 \\ 0'6877 ; & h = 0'25 \\ 0'69 ; & h = 0'125 \end{cases}$ and it is known that the exact solution is:

$y(t) = 3 - 0'1039 23t + 0'0009t^2$

errors:

$Y(30) = 3 - 0'103923 \cdot 30 + 0'0009 \cdot 30^2 = 0'69231$

$e(30) = \text{EXACT} - \text{APPROX}$

$h = 0'5 \rightarrow e_{0'5}(30) = 0'00921$; $h = 0'25 \rightarrow e_{0'25}(30) = 0'00461$; $h = 0'125 \rightarrow e_{0'125}(30) = 0'00231$

↳ The smaller the stepsize, the more accurate the result

$e_{0'25}/e_{0'5} \simeq 0'50054 \simeq 1/2 = 0'25/0'5$

→ they agree with the expected behaviour.

$e_{0'125}/e_{0'5} \simeq 0'50108 \simeq 1/2 = 0'125/0'25$

C) Estimate y(0'5) using pred. - corr. method ; h = 0'125

P $\underset{1}{y_{k+1}} = y_k + \frac{h}{2}(3\underset{1}{f_k} - \underset{0}{f_{k-1}})$; $\overset{C}{\underset{2}{y_{k+1}}} = \underset{1}{y_k} + \frac{h}{2}(\underset{2}{f_{k+1}} + \underset{1}{f_k})$ k=1

$Y(0) = 3$; $f = -0'06\sqrt{y}$

Y(0'25) = 2'974075549

$Y_{2P}(0'5) = \overset{t}{Y(0'25)} + \frac{0'25}{2}\left(3\overset{t}{f(0'25)} - \overset{t}{f(0)}\right)$ = 2'974075549 + 0'125 (-0'310419148 + 0'103923048) = 2'948263536

$Y_{2C1}(0'5) = \overset{t}{Y(0'25)} + 0'125\left(f(Y_{2P}) + \overset{t}{f(0'25)}\right)$ = 2'974075549 + 0'125 (-0'103023049 - 0'103473049) = 2'948263537

$\hookrightarrow \boxed{Y_{2C} \simeq 2'9483}$

Diff instability

- Decrease h (more nodes → more comp. cost)
- use an unconditionally stable method

4. $f''(z)$; $x_0 = z-2h$, $x_1 = z-h$, $x_2 = z+h$, $x_3\ z+2h$

A)

- $L_0(x) = \frac{(x-z+h)(x-z-h)(x-z-2h)}{(-h)(-3h)(-4h)} \rightarrow L_0'(x) = \frac{(x-z+h)(x-z-h)+(x-z+h)(x-z-2h)+(x-z-h)(x-z-2h)}{-12h^3}$

$\hookrightarrow L_0''(x) = \frac{2(x-z+h)+2(x-z-h)+2(x-z-2h)}{-12h^3} \rightarrow \boxed{L_0''(z) = \frac{\cancel{2h}-\cancel{2h}-4h}{-12h^3} = \frac{1}{3h^2}}$

- $L_1(x) = \frac{(x-z+2h)(x-z-h)(x-z-2h)}{(h)(-2h)(-3h)} \longrightarrow L_1''(x) = \frac{2(x-z+2h)+2(x-z-h)+2(x-z-2h)}{6h^3}$

$\hookrightarrow \boxed{L_1''(z) = \frac{2(\cancel{2h})+2(-h)+2\cancel{(-2h)}}{6h^3} = -\frac{1}{3h^2}}$

- $L_2(x) = \frac{(x-z+2h)(x-z+h)(x-z-2h)}{(3h)(2h)(-h)} \longrightarrow \boxed{L_2''(z) = \frac{4h+2h-4h}{-6h^3} = -\frac{1}{3h^2}}$

- $L_3(x) = \frac{(x-z+2h)(x-z+h)(x-z-h)}{(4h)(3h)(h)} \rightarrow \boxed{L_3''(z) = \frac{4h+\cancel{2h}-\cancel{2h}}{12h^3} = \frac{1}{3h^2}}$

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4.

A) $\boxed{D = \frac{1}{3h^2} \cdot f(z-2h) - \frac{1}{3h^2} f(z-h) - \frac{1}{3h^2} f(z+h) + \frac{1}{3h^2} f(z+2h) = \frac{f(z-2h) - f(z-h) - f(z+h) + f(z+2h)}{3h^2}}$

B) $E = -\frac{5}{12} f^{4}(z)h^2 - \frac{7}{120} f^{6}(z)h^4 - \frac{17}{4032} f^{(8)}(z)h^2 - \dots$ how would you obtain it?

- $E = e''(z) = \left(f[x_0, x_1, x_2, x_3, z]\, \Pi(z)\right)'' = \left(f'[x_0, x_1, x_2, x_3, z]\, \Pi(z) + f[x_0, x_1, \dots, z]\, \Pi'(z)\right)' =$

$= f''[x_0, x_1, x_2, x_3, z]\, \Pi(z) + 2f'[x_0, x_1, x_2, x_3, z]\, \Pi'(z) + f[x_0, x_1, x_2, x_3, z]\, \Pi''(z) = \dots$

- Anothe option is using Taylor's Σ

C) Justify what the pol. degree of exactitude is and the OC. Explain what they mean

- POLYNOMIAL DEGREE OF EXACTITUDE : N → our polynomial degree ≤ 3 and as we see the derivatives of f are of order ≥ 4 which mean they will be =0

$\hookrightarrow \boxed{N=3} \rightarrow E=0$

- ORDER OF CONVERGENCE : OC → The exponent of h of the principal term of E

↳ The lowest

$\hookrightarrow$ as $h \to 0$, $E \to 0$ $\qquad \hookrightarrow \boxed{OC=2}$

D) Considering $E = -\frac{5}{12} f^{(4)}(z) h^2$ hopt?

↳ UPPER BOUND

1.) $|E| = \left| \frac{-5}{12} f^{4}(z) h^2 \right| \leq \frac{5}{12} M \cdot h^2 = g_1(h)$

2) $AF = \sum_{i=0} |A_i| = 4/3h^2$

3) $|E_r| = \varepsilon / AF| = \frac{4\varepsilon}{3h^2} = g_2(h)$

4). $g(h) = g_1(h) + g_2(h) = \frac{5Mh^2}{12} + \frac{4\varepsilon}{3h^2}$

5) $g'(h) = 0 = \frac{5Mh}{6} - \frac{8\varepsilon}{3h^3}$

$\hookrightarrow \frac{5Mh}{6} = \frac{8\varepsilon}{3h^3}$

$h^4 = \frac{48\varepsilon}{15M} \rightarrow \boxed{h_{opt} = \sqrt[4]{\frac{48\varepsilon}{15M}} = 1'33748061 \sqrt[4]{\frac{\varepsilon}{M}}}$