ADVANCED NUMERICAL METHODS

BACHELOR DEGREE IN INDUSTRIAL TECHNOLOGY ENGINEERING

JUNE 30, 2016

SURNAMES: .... **Name:** .... **Group:** ……

Exercise 1	Exercise 2	Exercise 3	Exercise 4	***MARK***

N.B.: The exam will be sat uninterruptedly and will be marked over 42 points.

TIME: 3 hours

EXERCISE 1 **(13 points)**

Working with 5 significant digits:

A) To interpolate the function $f(x) = \text{Ln}(x)$ in the interval $[2, 3]$, 11 possible nodes are considered, namely, from $x = 2$ until $x = 3$ with separation $h = 0.1$. Calculate the polynomial $p_a(x)$ of degree ≤ 3 you expect to minimize the absolute-value truncation error at $x = 2.55$. (3p)

B) Calculate optimally the "exact" error (for the precision used) made when using $p_a(x)$ to interpolate the function $f(x)$ at point $x = 2.55$. Comment on the result obtained. (2p)

C) Estimate the truncation error of $p_a(x)$ at $x = 2.55$ using the value of $f(2.8)$. Can it be said that the value obtained will be a good approximation of that error? Reason out the answer. (2p)

D) Find an upper bound of the absolute-value truncation error of $p_a(x)$ $\forall\, x \in [2, 3]$. (2p)

E) Calculate the interpolation points for the polynomial $p_b(x)$ of degree ≤ 3 that interpolates $f(x)$ minimizing (approximately) the maximum absolute-value error over the interval $[2, 3]$. (2p)

F) Knowing that $p_b(2.55) = 0.93614$, compare the errors of $p_a(x)$ and $p_b(x)$ at $x = 2.55$. What can be concluded from this comparison? Reason out the answer. (0.5p)

G) Find an upper bound of the absolute-value truncation error of p_b in [2,3]. Compare with p_a. (1.5p)

EXERCISE 2 **(10 points)**

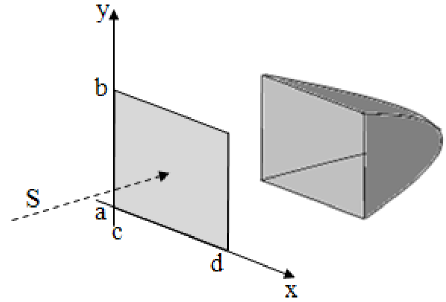
Consider $S = \int_a^b \int_c^d f(x, y)\, dx\, dy$, where S = torque, $f(x,y)$ = torsion function.

Estimate the torque producing the values in the table below on a 1.6 m × 1.2 m plate. Use the information in the table in the way you consider best so as to apply, operating with 3 decimal digits:

- The Trapezoidal rule for one of the variables.
- The second Simpson rule for the other one.

x \ y	0.0	0.2	0.4	0.6	0.8	1.0	1.2
0.0	0	0	0	0	0	0	0
0.4	0	3.123	4.794	5.319	4.794	3.123	0
0.8	0	3.818	5.960	6.647	5.960	3.818	0
1.2	0	3.123	4.794	5.319	4.794	3.123	0
1.6	0	0	0	0	0	0	0

Torsion function $f(x,y)$



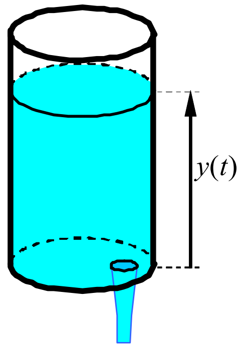
N.B.: Second Simpson rule:

$$\int_{u_0}^{u_3} f(u)\, du = \frac{3h}{8}\left[f(u_0) + 3f(u_1) + 3f(u_2) + f(u_3)\right] - \frac{3h^5}{80} f^{IV)}(\xi) \quad \text{for some } \xi \in (u_0, u_3)$$

EXERCISE 3 **(12 points)**

The emptying speed of a cylindrical tank after opening a valve located at its base is governed by the equation: $$y'(t) = -k\sqrt{y(t)}$$

where k is a constant that depends on the geometry of cylinder and valve.



A tank contains water initially up to 3 m high ($y(0) = 3$). Measuring y in m and t in minutes, the numeric value of the constant is $k = 0.06$.

A) Estimate the height of the liquid at $t = 0.5$ using the Midpoint method (Modified Euler) with step size $h = 0.25$. (3p)

B) Solving the problem with Euler's method, the following values for y at instant $t = 30$ are obtained:
- 0.6831 when the step size used is $h = 0.5$;
- 0.6877 when the step size used is $h = 0.25$;
- 0.6900 when the step size used is $h = 0.125$.

On the other hand, it is known that the exact solution to the problem is:

$$y(t) = 3 - 0.103923\,t + 0.0009\,t^2$$

Find the error made with those approximations and justify if what you observe agrees with the method's order of precision. (2p)

C) Estimate the value of $y(0.5)$ by means of a predictor-corrector method that uses the two formulas below with a P(EC)^{2}E scheme and step size $h = 0.25$. Use the Runge-Kutta method of order 2 of your choice to start the predictor-corrector one. (5p)

D) Explain briefly how you would detect in practice that a numerical method is being unstable while solving an initial-value problem, and how you would deal with that situation. (2p)

N.B.: Adams-Bashforth and Adams-Moulton methods of order 2:

$$y_{k+1} = y_k + \frac{h}{2}\left(3f_k - f_{k-1}\right); \qquad y_{k+1} = y_k + \frac{h}{2}\left(f_k + f_{k+1}\right)$$

EXERCISE 4 **(7 points)**

A) Find a numerical differentiation formula to estimate $f''(z)$ from the values of f at the points:

$$x_0 = z - 2h, \quad x_1 = z - h, \quad x_2 = z + h, \quad x_3 = z + 2h.$$ (2p)

B) The truncation error term E (or residue R) of the previous formula can be written in this way:

$$E = -\frac{5}{12} f^{4)}(z)h^2 - \frac{7}{120} f^{6)}(z)h^4 - \frac{17}{4032} f^{8)}(z)h^6 - \frac{341}{1814400} f^{10)}(z)h^8 + \ldots$$

Say briefly how you would obtain this expression if you had to. (1p)

C) Justify what the polynomial degree of exactitude and the order of convergence of that formula are, and explain briefly what each one of those two numbers means. (2p)

D) Considering only the principal term in E (the one of lowest degree), estimate the optimal step size h_{opt} of the formula, and define briefly the parameters you have written your estimation in terms of. (2p)