ADVANCED NUMERICAL METHODS

DEGREE IN INDUSTRIAL TECHNOLOGY ENGINEERING

JULY 5, 2014

N.B. The exam will be sat uninterruptedly (without a middle break), and it will be marked over 35 points.

1.- The following data points come from some function $f(x)$:

x_i	0	1	1.5	11	11.5	12.5
y_i	0	1	0	0	1	0

Working with 3-significant-digit arithmetic throughout:

a) Calculate the table of differences. **(1.5p)**

b) Write the Newton polynomial $p_5(x)$, and indicate its degree. Is there any other polynomial that also passes by these points? Of what degree? **(1.5p)**

c) In what senses is Hörner's algorithm to evaluate polynomials optimal? **(0.5p)**

d) Evaluate $p_5(6.8)$ optimally. **(1p)**

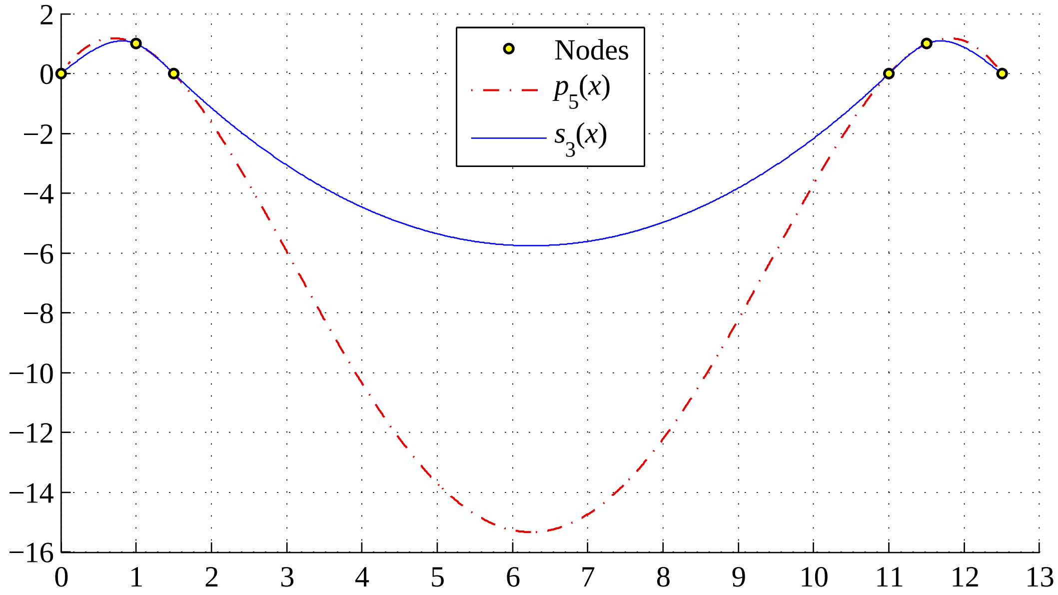
e) Write $p_5(x)$ in terms of Lagrange base functions, and evaluate it on $x = 6.8$. Knowing that $p_5(6.8) = -15.0052...$, does anything in the result surprise you? **(1.5p)**

f) If you had two new data points of $f(x)$, describe the calculations you would make to estimate the error made when approximating $f(6.8)$ by $p_5(6.8)$. **(1.5p)**

g) If $f(6.7) = 0$ and $f(6.9) = 0$, estimate that error without calculating anything. **(0.5p)**

2.- a) Explain in detail in what sense natural cubic splines are optimal. **(2p)**

b) Comment the previous result in relation with the following figure, which shows the interpolation polynomial $p_5(x)$ and the natural cubic spline $s_3(x)$ by the data in the initial table of the previous exercise: **(1p)**



3.- *Gauss's error function* (or simply the *error function*) is a special function (i.e. non-elementary) that appears in probability, statistics, partial differential equations, etc., and it is defined as follows:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

One wants to approximate erf(0.3) using the compound open Newton-Cotes rule of two nodes.

- **a)** Calculate the weights of the simple rule in terms of the distance h between nodes. **(1p)**
- **b)** Calculate the error term of the simple rule. **(1p)**
- **c)** Calculate the error term of the compound rule, justifying the steps taken. **(1p)**
- **d)** Knowing that the second derivative of $\exp(-t^2)$ is increasing in [0,0.3], how many subintervals N guarantee that the absolute error is less than $5\cdot 10^{-4}$? **(1p)**
- **e)** Calculate with 5 decimals the result obtained with that number of subintervals. **(1p)**
- **f)** Find bounds of the error made in the previous section and, knowing that the exact value is $\operatorname{erf}(0.3) = 0.32862675946...$, check that the bounds are satisfied. **(1p)**
- **g)** From the exact value and the method's order of convergence, estimate the result one would expect to obtain with $N = 4$ subintervals. **(1p)**
- **h)** The IEEE 754 standard of double-precision arithmetic handles about 16 decimal significant digits (not exactly 16 because the operations are carried out in base 2). Calculate how many subintervals N would be required to guarantee an absolute error less than 10^{-16}, and draw some practical conclusion from the result. **(1p)**

4.-

- **a)** Derive the expression of the amplification factor of the roundoff error in a numerical differentiation formula to calculate $f^{k)}(z)$. **(4p)**
- **b)** Obtain a formula to approximate the value of $f''(z)$ with the highest possible precision from the information in $f(z)$, $f(z+2h)$ and $f(z-2h)$. **(2p)**
- **c)** Obtain its error term. Of what order is the formula obtained? **(2p)**
- **d)** Determine the optimal size h of the formula obtained for the function $f(x) = \sin(x) + \cos(x)$. **(2p)**

(There is one more sheet of paper.)

5.- a) Check that the following problem has a unique solution in the interval [0,0.5]:

$$\begin{cases} y' = (1+t)y \\ y(0) = 1 \end{cases}$$

(1p)

b) Using the values in the following table:

x_i	y_i	f_i
0	1	1
0.1	1.11071	1.22178
0.2	1.24608	1.49530
0.3	1.41199	1.83559

apply Adam's predictor-corrector method to obtain two more points of the solution of the problem with $h = 0.1$. At every step carry out the number of iterations necessary to obtain a precision 10^{-4} in the calculation of the solution using a scheme $P(EC)^S E$. Operate with roundoff to 5 decimals.

Predictions: Adams-Bashforth of 4 steps:

$$y_{n+1} = y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right]$$

Corrections: Adams-Moulton of 3 steps:

$$y_{n+1} = y_n + \frac{h}{24}\left[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right]$$

(5p)

TOTAL TIME: 3 hours