ADVANCED NUMERICAL METHODS

3rd year Bachelor Degree in Industrial Technology Engineering, 7/3/2013

PART 1

• Time: 1 hour and 45 minutes. (After a brief break, Part 2 will start.)

EXERCISE 1:

A) Prove that the interpolation polynomial of degree ≤ 3 with 4 distinct nodes exists and is unique. (3p)

B) Calculate the table of differences corresponding to the following conditions:

$f(0)=1;\quad f(1)=4;\quad f(2)=1;\quad f(3)=4;\quad f'(1)=-2;\quad f''(1)=-6;\quad f'(3)=10$ (2.5p)

C) Approximate $f'''(x)$ without calculating any derivative, *stating* the result used to do it. (1.5p)

D) Calculate the corresponding osculating polynomial $p(x)$ and evaluate $p(0.25)$ optimally. (2.5p)

E) Does there exist any other polynomial of the same degree as $p(x)$, or of lower degree, that also satisfies the conditions above? Justify the answer. (1p)

F) Consider the following polynomial:

$$p_7(x) = x^7 - 11x^6 + 48x^5 - 106x^4 + 127x^3 - 84x^2 + 28x + 1$$

Without calculating anything, can you discard that it satisfies the conditions above? If you can, explain why; otherwise, describe how you could find other polynomials of degree 7 that also satisfy them. (1.5p)

EXERCISE 2: **(Use roundoff to 4 significant digits)**

A) From the Legendre polynomial $\varphi_4(x) = (35/8)x^4 - (15/4)x^2 + 3/8$, calculate the nodes of the Gauss-Legendre quadrature rule of four points, indicating up to what degree it will integrate polynomials exactly. (1.5p)

B) Write, without solving it, a linear system of equations to obtain the coefficients of the formula, and find its solution in the Appendix on the back. (1.5p)

C) Calculate the formula's error term. (2p)

One wants to approximate $I = \int_{-1}^{3} f(x)dx$ without evaluating $f(x)$ more than 8 times.

D) Using the previous results only, what nodes would you choose in $[-1,3]$, and what coefficients? (N.B. Use the compound formula adding the simple formulas in $[-1,1]$ and in $[1,3]$.) (2p)

E) Find bounds of the error of the previous formula for $f(x)=e^{x/2}$. Knowing that the application of the formula yields the result 7.750316817, check if it stays within the bounds obtained. (2p)

F) If you could now use all the information in the Appendix, what nodes and what coefficients would you choose in $[-1,3]$? (1p)

Appendix:

Gauss-Legendre quadrature:

n	Nodos	Coeficientes	Término de error
0	0	2	$(1/3)\, f''(\xi)$
1	±0.57735 02692	1.00000 00000	$(1/135)\, f^{4)}(\xi)$
2	0.00000 00000 ±0.77459 66692	0.88888 88889 0.55555 55556	$\frac{1}{15750} f^{6)}(\xi)$
3	±0.33998 10436 ±0.86113 63116	0.65214 51549 0.34785 48451	$\frac{1}{3472875} f^{8)}(\xi)$
4	0.00000 00000 ±0.53846 93101 ±0.90617 98459	0.56888 88889 0.47862 86705 0.23692 68851	$\frac{2^{11}\times 5!^4}{11\times 10!^3} f^{10)}(\xi)$
5	±0.23861 91861 ±0.66120 93865 ±0.93246 95142	0.46791 39346 0.36076 15730 0.17132 44924	$\frac{2^{13}\times 6!^4}{13\times 12!^3} f^{12)}(\xi)$
6	0.00000 00000 ±0.40584 51514 ±0.74153 11856 ±0.94910 79123	0.41795 91837 0.38183 00505 0.27970 53915 0.12948 49662	$\frac{2^{15}\times 7!^4}{15\times 14!^3} f^{14)}(\xi)$
7	±0.18343 46425 ±0.52553 24099 ±0.79666 64774 ±0.96028 98565	0.36268 37834 0.31370 66458 0.22238 10345 0.10122 85363	$\frac{2^{17}\times 8!^4}{17\times 16!^3} f^{16)}(\xi)$
8	0.00000 00000 ±0.32425 34234 ±0.61337 14327 ±0.83603 11073 ±0.96816 02395	0.33023 93550 0.31234 70770 0.26061 06964 0.18064 81607 0.08127 43884	$\frac{2^{19}\times 9!^4}{19\times 18!^3} f^{18)}(\xi)$

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PART 2

• Time: 1 hour and 15 minutes.

EXERCISE 3:

A) Give the general expression of the truncation error of a numerical differentiation formula (first derivative). Analyzing that expression, say in which simple cases one can obtain different formulas of numerical differentiation. (2p)

B) For $n=1$, indicate three possible placements of z with respect to the nodes, and derive, among the three of them, the one of highest order. (2p)

C) Write a MATLAB function implementing that formula.
(The explanations can be given as brief comments within the code itself.) (2p)

EXERCISE 4:

A) Find A_0, A_1, A_2 so that the following numerical differentiation formula is of interpolatory kind:

$$f'(1) = A_0 f(-1) + A_1 f(0) + A_2 f(1) + R[f] \qquad (2p)$$

B) Obtain the expression of the truncation error. (2p)

EXERCISE 5:

Given the ordinary differential equation $y'' + 42y' + 40y = 0$
subject to the initial conditions $y(0)=1, \quad y'(0)=0.$

A) Transform the equation into a system of order one. (1p)

B) Apply the Runge-Kutta method of order 2 (Enhanced Euler) with step size $h=0.1$ to approximate $y(0.2)$ and $y'(0.2)$. (3p)

EXERCISE 6:

Solve the following initial value problem by means of the Milne predictor-corrector method with step size $h=0.5$:

$$\begin{cases} y' = 4\sin(0.8x) - 0.5\cos(y) \\ y(0) = 1 \end{cases} \qquad x \in [0,2]$$

using as approximations $y(0.5) \approx 1.27442$, $y(1) \approx 2.44526$, $y(1.5) \approx 4.32615$. Operate with roundoff to 6 significant digits and use a $PE(CE)^2$ scheme. Estimate the relative error made.

Predictor: $$y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2f_n - f_{n-1} + 2f_{n-2}\right] \quad (n = 3, \ldots, N-1)$$

Corrector: $$y_{n+1} = y_{n-1} + \frac{h}{3}\left[f_{n+1} + 4f_n + f_{n-1}\right] \quad (n = 1, 2, \ldots, N-1) \qquad (4p)$$